

## Research Article

### Significance Test of Reliability Evaluation with Three-parameter Weibull Distribution Based on Grey Relational Analysis

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**Abstract:** With the aid of the grey system theory, the grey relational analysis of the reliability with the three-parameter Weibull distribution is made for the Weibull parameter evaluation and its significance test. Via the theoretical value set and the experimental value set of the reliability relied on the lifetime data of a product, the model of the constrained optimization of the Weibull parameter evaluation based on the maximum grey relational grade. The grey significance of the reliability function with the three-parameter Weibull distribution is tested by means of the proposed criterion of the grey significance analysis of the reliability evaluation at the given grey confidence level. The cases of the helicopter component, the specimen and the ceramic material show that the grey relational analysis of the reliability is effective in the Weibull parameter evaluation and its significance test.

**Keywords:** Grey relational analysis, parameter evaluation, reliability, product, significance test, three-parameter weibull distribution

## INTRODUCTION

For many products, such as spacecrafts, airplanes, bullet trains, nuclear reactors and submarines, the reliability is regarded as an important indicator for their safe and stable operation. In reliability analyses, the three-parameter Weibull distribution as a classical probability distribution is widely used in many scientific studies and engineering practices.

For example, Duffy *et al.* (1993a) described nonlinear regression estimators for the three-parameter Weibull distribution and made a reliability analysis of a turbopump blade; Ichikawa (1992) reviewed application of the three-parameter Weibull distribution in relation to reliability evaluation of ceramic components; Tsionas (2000) developed Bayesian analysis in the context of samples from three-parameter Weibull distributions and shown how to tackle the problems of prediction and estimation of reliability curves; Sürücü and Sazak (2009) considered a three-parameter Weibull distribution to model inter-failure times and used a robust estimation technique to estimate the unknown parameters; Bai and Mu (2011) used the three-parameters Weibull distribution to process the failure data of belt transportation system and presented the method of dynamic reliability evaluating of belt transportation system in mines by the practical example of the data statistical analysis; Ren *et al.* (2012) established the three-parameter Weibull distribution model of time between faults to improve the model

accuracy of reliability evaluation of the machining center.

It can be seen that the primary concern in reliability analysis is to evaluate the Weibull parameters and so far many good results are achieved. But, from a standpoint of the statistical theory (Al-Rawi and Silva, 2012; Vetrova and Bardsley, 2012; Xia *et al.*, 2012), the result of the parameter evaluation should follow a confidence level, that is, needs to conduct the significant test based on the fact that many complex and variable factors appear in life tests and the result may therefore be distorted, resulting in hidden dangers. For this reason, this study proposes a method for significance test of the reliability evaluation with the three-parameter Weibull distribution based on the grey relational analysis.

## THEORETICAL VALUE SET AND EXPERIMENTAL VALUE SET OF RELIABILITY

The three-parameter Weibull distribution is defined as:

$$f(t; \eta, \beta, \tau) = \frac{\beta}{\eta} \left( \frac{t - \tau}{\eta} \right)^{\beta - 1} \exp \left( - \left( \frac{t - \tau}{\eta} \right)^{\beta} \right) \quad (1)$$
$$t \geq \tau > 0; \beta > 0; \eta > 0$$

where,

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$f(t; \eta, \beta, \tau)$  : The three-parameter Weibull distribution  
 $t$  : A stochastic variable of the lifetime  
 $(\eta, \beta, \tau)$  : The mark of the Weibull parameters  
 $\eta$  : The shape parameter  
 $\beta$  : The scale parameter  
 $\tau$  : The location parameter

The reliability function with the three-parameter Weibull distribution can be expressed as:

$$R(t; \eta, \beta, \tau) = \exp\left(-\left(\frac{t-\tau}{\eta}\right)^\beta\right) \quad (2)$$

where,

$R(t; \eta, \beta$  and  $\tau)$  : The reliability function with the three-parameter Weibull distribution

For many products in engineering practices, the Weibull parameter ( $\eta, \beta,$  and  $\tau$ ) are unknown and need to be found out with the help of test evaluation. Thus, a lifetime experiment must be conducted. Assume that the lifetime data of a product is obtained as:

$$T = \{t_i\}; t_1 \leq t_2 \leq \dots \leq t_i \leq \dots \leq t_n; i = 1, 2, \dots, n \quad (3)$$

where,

$T$  = The lifetime data set

$t_i$  = The  $i^{\text{th}}$  lifetime datum in  $T$

$i$  = The sequence number of  $t_i$

$n$  = The number of the data in  $T$

The lifetime data in  $T$  are substituted into Eq. (2) and the theoretical value of the reliability can be calculated by:

$$\mathbf{R}_0 = \{R(t_i; \eta, \beta, \tau)\} \quad (4)$$

where,

$R(t_i; \eta, \beta, \tau)$  : The theoretical value of the reliability

$\mathbf{R}_0$  : The theoretical value set of the reliability

Under the condition of unknown Weibull parameters, the Nelson method (Nelson, 1990) and the Johnson method (Johnson, 1970) can be employed to make a nonparametric estimation for the empirical values of the reliability, as follows:

$$r = r(\eta, \beta, \tau) = \frac{1}{n} \sum_{i=1}^n \frac{\min_i |r(t_i) - R(t_i; \eta, \beta, \tau)| + \xi_0 \max_i |r(t_i) - R(t_i; \eta, \beta, \tau)|}{|r(t_i) - R(t_i; \eta, \beta, \tau)| + \xi_0 \max_i |r(t_i) - R(t_i; \eta, \beta, \tau)|} \quad (8)$$

where,

$r \in (0, 1)$  : The grey relational grade of the experimental value set  $\mathbf{R}_1$  to the theoretical value set  $\mathbf{R}_0$

$\xi_0 \in (0, 1)$  : The distinguishing coefficient given in advance

$$\mathbf{R}_1 = \{r(t_i)\} \quad (5)$$

where,

$r(t_i)$  : Called the  $i^{\text{th}}$  empirical value of the reliability

$\mathbf{R}_1$  : The empirical value set of the reliability

According to the Johnson method (Johnson, 1970), the empirical value of the reliability is expressed as an expectation, as follows:

$$r(t_i) = 1 - \frac{i}{n+1}; i = 1, 2, \dots, n \quad (6)$$

According to the Nelson method (Nelson, 1990), the empirical value of the reliability is expressed as a median, as follows:

$$r(t_i) = 1 - \frac{i - 0.3}{n + 0.4} \quad (7)$$

In reliability evaluation, the empirical value  $r(t_i)$  in  $\mathbf{R}_1$  can be considered as the experimental value of the reliability due to its corresponding to and depending on the lifetime datum  $t_i$  in  $T$  and  $\mathbf{R}_1$  can accordingly be called the experimental value set of the reliability.

With this, two sets,  $\mathbf{R}_0$  and  $\mathbf{R}_1$ , that are, respectively, the theoretical value set and the experimental value set, are obtained and they can be used for an estimation of the three Weibull parameters by means of the grey relational grade.

**Weibull parameter evaluation:** The systems can be divided into three types: the black system, the white system and the grey system. The black system is not perceived one, the white system is perceived one and the grey system is a partly perceived and unascertained system between the black and white systems.

The grey relational grade as a very important concept in the grey system theory is widely used in many fields of science and technology (Xia *et al.*, 2012, 2010; Wen *et al.*, 2011; Sridhar *et al.*, 2012; Palanikumar *et al.*, 2012). For example, it can be employed to evaluate the difference of the attributes of systems according to the geometry shape of data series. The bigger the value the grey relational grade takes, the smaller the difference of the attributes of systems; or else, the bigger the difference. The grey relational grade is given by:

The most common distinguishing coefficient  $\xi_0$  is 0.5 in the Weibull parameter evaluation based on the maximum grey relational grade.

The optimal estimation of the Weibull parameter  $(\eta, \beta, \tau)$  should makes the grey relational grade  $r$  maximum, as follows:

$$r \rightarrow r_{\max} = \max_{(\eta, \beta, \tau)} r(\eta, \beta, \tau) \tag{9}$$

The constraint conditions in Eq. (9) are given by:

$$0 < \tau \leq t_1 \tag{10}$$

$$\eta > 0 \tag{11}$$

$$\beta > 0 \tag{12}$$

Equation (9) to (12) is called the model of the constrained optimization of Weibull parameter evaluation based on the maximum grey relational grade. With the help of the model, the optimal estimation for the Weibull parameter can be obtained as:

$$(\eta, \beta, \tau) = (\eta^*, \beta^*, \tau^*) \tag{13}$$

where,

$(\eta^*, \beta^*, \tau^*)$  : The optimal estimation for  $(\eta, \beta, \tau)$

The model means that the experimental value set  $R_1$  is approaching to the theoretical value set  $R_0$  along with the optimal estimation  $(\eta^*, \beta^*, \tau^*)$  of the Weibull parameter  $(\eta, \beta, \tau)$ , when the grey relational grade  $r$  is approaching to the maximum grey relational grade  $r_{\max}$ .

### GREY SIGNIFICANCE TEST OF RELIABILITY EVALUATION

Considered the first element  $R(t_1; \eta^*, \beta^*, \tau^*)$  in  $R_0$  as the reference value, two grey relational grades are defined as:

$$\gamma_0(\xi) = \frac{1}{n} \sum_{i=1}^n \frac{\min_i |R(t_i; \eta^*, \beta^*, \tau^*) - R(t_1; \eta^*, \beta^*, \tau^*)| + \xi \max_i |R(t_i; \eta^*, \beta^*, \tau^*) - R(t_1; \eta^*, \beta^*, \tau^*)|}{|R(t_i; \eta^*, \beta^*, \tau^*) - R(t_1; \eta^*, \beta^*, \tau^*)| + \xi \max_i |R(t_i; \eta^*, \beta^*, \tau^*) - R(t_1; \eta^*, \beta^*, \tau^*)|} \tag{14}$$

and

$$\gamma_1(\xi) = \frac{1}{n} \sum_{i=1}^n \frac{\min_i |r(t_i) - R(t_1; \eta^*, \beta^*, \tau^*)| + \xi \max_i |r(t_i) - R(t_1; \eta^*, \beta^*, \tau^*)|}{|r(t_i) - R(t_1; \eta^*, \beta^*, \tau^*)| + \xi \max_i |r(t_i) - R(t_1; \eta^*, \beta^*, \tau^*)|} \tag{15}$$

where,

$\gamma_0(\xi)$  : The grey relational grade of  $R(t_i; \eta^*, \beta^*, \tau^*)$  to  $R(t_1; \eta^*, \beta^*, \tau^*)$

$\gamma_1(\xi)$  : The grey relational grade of  $r(t_i)$  to  $R(t_1; \eta^*, \beta^*, \tau^*)$

$\xi \in (0, 1)$  : The distinguishing coefficient to be solved in the grey significance analysis of the reliability evaluation

$R(t_1; \eta^*, \beta^*, \tau^*)$  : The reference value

The grey difference is defined as:

$$d(\xi) = |\gamma_1(\xi) - \gamma_0(\xi)| \tag{16}$$

The grey difference can be employed to describe the difference degree between  $R_0$  and  $R_1$ . Clearly, the grey difference varies with the distinguishing coefficient. The maximum grey difference can accordingly be given by:

$$d^* = \max_{\xi = \xi^*} d(\xi) \quad (17)$$

where

$\xi^*$ : The optimum distinguishing coefficient

The equivalence coefficient of  $R_0$  and  $R_1$  is defined as:

$$f = \begin{cases} 1 - \frac{d^*}{\eta}; & d^* \in (0, \eta) \\ 0 & ; d^* \in [\eta, 1] \end{cases} \quad (18)$$

where,

$f \in [0, 1]$ : The equivalence coefficient of  $R_0$  and  $R_1$

$\eta \in (0, 1]$ : The weight

Let the grey significance level  $\alpha \in [0, 1]$  and then grey confidence level can be given by:

$$P_G = (1 - \alpha) \times 100\% \quad (19)$$

The most common grey significance levels are 0.05 in the grey significance analysis of the reliability evaluation based on the maximum grey relational grade.

The relation of the grey confidence level with the weight is:

$$\eta = \frac{1}{\lambda} \left( 1 - \frac{P_G}{100} \right) \quad (20)$$

where,

$\lambda \in [0, 1]$ : The level that also is called a cut of the relation between  $R_0$  and  $R_1$

The grey confidence level  $P_G$  is able to describe the reliability of the relation between  $R_0$  and  $R_1$  and the criterion of the grey significance analysis of the reliability evaluation is given below.

**Criterion:** if

$$f \geq \lambda \quad (21)$$

Then the relation between  $R_0$  and  $R_1$  is close, meaning that the result of the reliability evaluation is of significance at the  $P_G$  grey confidence level; otherwise the relation between  $R_0$  and  $R_1$  is not close, meaning

that the result of the reliability evaluation is of no significance at the  $P_G$  grey confidence level.

The significance of the reliability function with the three-parameter Weibull distribution can be tested with the help of the given grey confidence level and the criterion of the grey significance analysis of the reliability evaluation. If the result of the reliability evaluation is of significance at the  $P_G$  grey confidence level, the reliability function  $R(t; \eta^*, \beta^*, \tau^*)$  with the three-parameter Weibull distribution is of value in the engineering practice studied; or else, the reliability function  $R(t; \eta^*, \beta^*, \tau^*)$  with the three-parameter Weibull distribution is of no value in the engineering practice studied.

In the light of the least information principle in the grey system theory, the optimum level is  $\lambda = 0.5$ .

## CASE STUDIES

**Case 1:** This is a case of the helicopter component. In failure analysis, the lifetime data set of a helicopter component is obtained, as follows ( $n = 13$ ) (Luxhoy and Shyur, 1995):

$$T = (156.5, 213.4, 265.7, 265.7, 337.7, 337.7, 406.3, 573.5, 573.5, 644.6, 744.8, 774.8, 1023.6)$$

For the Weibull parameter evaluation, the results are  $(\eta^*, \beta^*, \tau^*) = (451.9850, 1.6561, 97.2279)$  and  $r_{\max} = 0.7682$ . The diagram of  $R_0$  and  $R_1$  (the median of the reliability) are presented in Fig. 1. Overall, the trend of the experimental value set  $R_1$  is in very good accordance with that of the theoretical value set  $R_0$ , but its uncertainty and fluctuation relative to  $R_0$  is big.

For the grey significance test of the reliability evaluation, let  $\alpha = 0.05$  and  $\lambda = 0.5$ , then the results are  $P_G = 95\%$  and  $f = 0.234 < 0.5$ . From the criterion, the reliability function  $R(t; \eta^*, \beta^*, \tau^*)$  with the three-parameter Weibull distribution is of no value in the engineering practice studied at the 95% grey confidence level (this likely roots in the big uncertainty and fluctuation of  $R_1$  relative to  $R_0$  as shown in Fig. 1).

**Case 2:** This is a case of some specimen. Under the same level of stress the fatigue life data set of the specimen is obtained as ( $n = 20$ ) (Deng *et al.*, 2004):

$$T = (350, 380, 400, 430, 450, 470, 480, 500, 520, 540, 550, 570, 600, 610, 630, 650, 670, 730, 770, 840)$$

For the Weibull parameter evaluation, the results are  $(\eta^*, \beta^*, \tau^*) = (338.2694, 2.2505, 259.5402)$  and  $r_{\max} = 0.9486$ . The diagram of  $R_0$  and  $R_1$  (the expectation of the reliability) are presented in Fig. 2. Clearly, the experimental value set  $R_1$  is in very good accordance with the theoretical value set  $R_0$  and its uncertainty and fluctuation relative to  $R_0$  is very small.

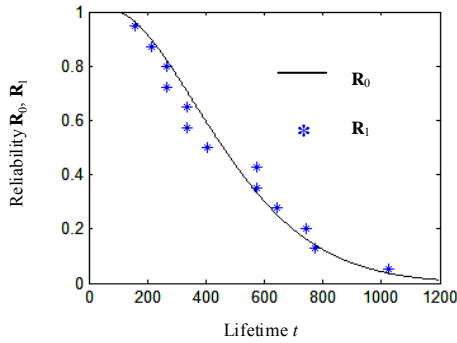


Fig. 1: Theoretical value set  $R_0$  and experimental value set  $R_1$  of reliability in case 1

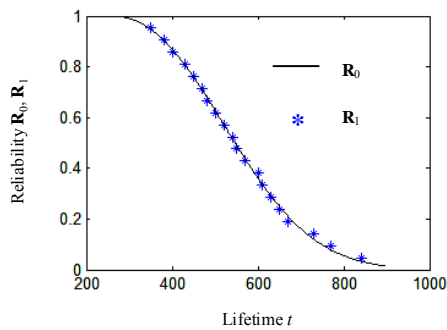


Fig. 2: Theoretical value set  $R_0$  and experimental value set  $R_1$  of reliability in case 2

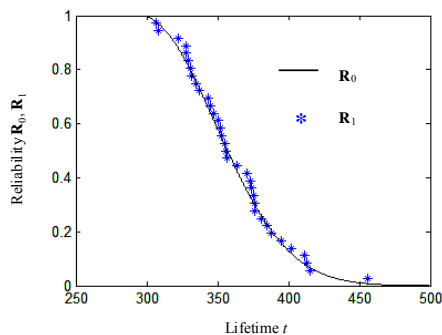


Fig. 3: Theoretical value set  $R_0$  and experimental value set  $R_1$  of reliability in case 1

For the grey significance testing of the reliability evaluation, let  $\alpha = 0.05$  and  $\lambda = 0.5$ , then the results are  $P_G = 95\%$  and  $f = 0.519 > 0.5$ . From the criterion, the reliability function  $R(t; \eta^*, \beta^*, \tau^*)$  with the three-parameter Weibull distribution is of value in the engineering practice studied at the 95% grey confidence level (this likely roots in the very small uncertainty and fluctuation of  $R_1$  relative to  $R_0$  as shown in Fig. 2).

**Case 3:** This is a case of the ceramic material. In life tests, Duffy *et al.* (1993b) obtained the lifetime data set of aluminum oxide ceramics, as follows ( $n = 35$ ):

$T = (307, 308, 322, 328, 328, 329, 331, 332, 335, 337, 343, 345, 347, 350, 352, 353, 355, 356, 357, 364, 371, 373, 374, 375, 376, 376, 381, 385, 388, 395, 402, 411, 413, 415, 456)$

For the Weibull parameter evaluation, the results are  $(\eta^*, \beta^*, \tau^*) = (73.7708, 2.0351, 295.3628)$  and  $r_{max} = 0.7296$ . The diagram of  $R_0$  and  $R_1$  (the expectation of the reliability) are presented in Fig. 3. It is not hard to see that the experimental value set  $R_1$  is in very good accordance with the theoretical value set  $R_0$  and its uncertainty and fluctuation relative to  $R_0$  is very small.

For the grey significance test of the reliability evaluation, let  $\alpha = 0.05$  and  $\lambda = 0.5$ , then the results are  $P_G = 95\%$  and  $f = 0.714 > 0.5$ . From the criterion, the reliability function  $R(t; \eta^*, \beta^*, \tau^*)$  with the three-parameter Weibull distribution is of value in the engineering practice studied at the 95% grey confidence level (this likely roots in the very small uncertainty and fluctuation of  $R_1$  relative to  $R_0$  as shown in Fig. 3).

Three case studies above show that grey relational analysis of the reliability with the three-parameter Weibull distribution is effective in the Weibull parameter evaluation and its significance test.

## CONCLUSION

The model of the Weibull parameter evaluation based on the maximum grey relational grade means that the experimental value set of the reliability is approaching to the theoretical value set of the reliability along with the optimal estimation of the Weibull parameters, when the grey relational grade is approaching to the maximum grey relational grade.

The significance of the reliability function with the three-parameter Weibull distribution can be tested with the help of the given grey confidence level and the criterion of the grey significance analysis of the reliability evaluation.

The cases of the helicopter component, the specimen and the ceramic material show that the grey relational analysis of the reliability with the three-parameter Weibull distribution is effective in the Weibull parameter evaluation and its significance test.

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