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Research Article

A Comprehensive Comparison between Wave Propagation and Heat Distribution via Analytical Solutions and Computer Simulations

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Abstract: Wave propagation and heat distribution are both governed by second order linear constant coefficient partial differential equations, however their solutions yields very different properties. This study presents a comprehensive comparison between hyperbolic wave equation and parabolic heat equation. Issues such as conservation of wave profile versus averaging, transporting information, finite versus infinite speed propagation, time reversibility versus irreversibility and propagation of singularities versus instantaneous smoothing have been addressed and followed by examples and graphical evidences from computer simulations to support the arguments.

Keywords: Heat distribution, hyperbolic and parabolic partial differential equation, wave propagation

INTRODUCTION

Wave propagation is described by hyperbolic Partial Differential Equation (PDE) which is derived using the dynamic equilibrium law (Graff, 1991) and can be written in three dimensions as: $u_{tt} = c^2\nabla^2 u$, where c is a fixed positive constant and ∇^2 is the Laplacian operator. Motion of a vibrating string is an example of wave equation. Heat distribution is described by heat equation which is a parabolic PDE. It is obtained through simultaneously applying of the Fourier law (Divo and Kassab, 2002) and the energy conservation law (Cannon, 1984) and is written in three dimensions as: $u_t = \beta\nabla^2 u$, where β is a constant and is related to material property (Incropera and DeWitt, 2002).

Both hyperbolic and parabolic PDE's appear in the modeling of different natural phenomena and has practical applications in engineering problems (Solin, 2005). For example Unsworth and Duarte (1979) proposed dual theoretical experimental method for measurement of the thermal diffusivity in polymers that is applicable to rubber and various other materials. A collection of comprehensive studies on PDEs application, wave theory and diffusion process can be found in the works of Ozisik (1993), Asmar (2004), Wazwaz (2009) and Thambynayagam (2011). Other specific areas where heat equation has been widely used include image analysis (Perona and Malik, 1990), in machine learning as the motivating theory behind Laplacian methods and in financial mathematics in the modeling of options (Thambynayagam, 2011). Complicated mathematical model's differential equation can also be transformed into PDEs, for example, the famous Black-Scholes option pricing model's differential equation can be transformed into the heat equation allowing relatively easy solutions from a familiar body of mathematics (MacKenzie, 2006).

One of the several analytical approaches for solving the governing equations of wave propagation and heat distribution containing time derivative terms is the Fourier method (Polyanin, 2002). The general solution to the one dimensional wave equation is given by d´Alembert's formula. Numerous researchers have studied fundamental solution of wave propagation and heat transfer in homogenous and non-homogenous media. Rizos and Zhou (2006) applied this method to solve wave propagation problem in three dimensional media. Young *et al*. (2004) studied fundamental solution of heat transfer in homogenous media. Variational iteration method was used by Yulita *et al*. (2009) for fractional heat and wave-like equations. Momani (2005) applied analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method. Batiha *et al*. (2007) studied application of variational iteration method to heat and wave-like equations. An integral operator mapping functions approach was applied by Gzyl (1992) to solve wave equations via the solution to heat equations. Lam and Fong (2001) used analytical solution to perform a study on heat diffusion

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versus wave propagation in solids subjected to exponentially-decaying heat source. Wazwaz and Gorguis (2004) used exact solutions for heat-like and wave-like equations with variable coefficients. A comparison of the solutions of a phase-lagging heat transport equation and damped wave equation has been discussed in the work of Su *et al*. (2005).

The objective of this study is to present a comprehensive comparison between wave and heat equations. The governing equations of wave propagation and heat distribution are first recalled through practical examples and their solutions are discussed for the purpose of comparing solution behavior. Computer simulations and graphical results are then provided as evidence to support the mathematical arguments.

METHODOLOGY

Wave propagation model: We consider the following Initial Value Problem (IVP) describing the vertical displacement (at time t and position x) of an infinitely long, perfectly flexible, homogenous string, stretched along the horizontal x −axis, in the absence of external forces, with mass density ρ and tension coefficient τ:

$$u_{tt} = c^2 u_{xx} \ for \ t \in \mathbb{R} \ and \ x \in \mathbb{R}, \quad u(x,0) = f(x), \ u_t(x,0) = g(x) \ for \ x \in \mathbb{R} \tag{1}$$

where, $c = \tau/\rho$ is a positive constant and $f, g : \mathbb{R} \to \mathbb{R}$ are given functions, representing the initial displacement and velocity of the string. This is an example of a standing wave (a wave that vibrates in place without lateral motion along the string). The boundary conditions will dictate the condition of the string at the ending points, (whether it is held fixed or not). The initial conditions usually specify the initial displacement and the initial velocity of each point. We first show that the unique solution of this IVP is given by:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x+ct}^{x-ct} g(s)ds \tag{2}$$

If we let $\mu = x - ct$ and $\eta = x + ct$ and then use the chain rule, we will get: $\partial/\partial x = \partial/\partial\mu + \partial/\partial\eta$ and $\partial/\partial t = -c\,\partial/\partial\mu + c\,\partial/\partial\eta$. Plugging these into (1) yields: $u_{xx} = (\partial^2 u)/(\partial\mu^2) + 2\,(\partial^2 u)/\partial\eta\partial\mu + (\partial^2 u)/(\partial\mu^2)$ and $(1/c^2)\,u_{tt} = (\partial^2 u)/(\partial\mu^2) - 2(\partial^2 u)\,/\,\partial\eta\partial\mu + (\partial^2 u)/\,(\partial\mu^2)$ which indicates $\partial^2 u/\partial\eta\partial\mu = 0$, leading to the general solution of the form: $u(x,t) = \phi(x+ct) + \psi(x-ct)$, where $\phi, \psi: \mathbb{R} \to \mathbb{R}$ are arbitrary twice differentiable functions. The initial conditions in (1) imply and necessitate that $f(x) = \phi(x) + \psi(x)$ and $\left(\frac{1}{c}\right) G(x) = \phi(x) - \psi(x)$ where $G(x)$ is an anti-derivative of $g(x)$. The solution in (2) defines a continuous function $u = u(x,t)$ as long as f is continuous and g is integrable. Under these assumptions, the partial derivatives u_{tt} and u_{xx} may not exist everywhere, but u may still be considered a weak solution of the IVP (1). In fact, we now need two functions f and G to be twice differentiable (that is, f must be twice and g at least once differentiable). If we let the initial conditions: $u(x,0) = f(x) = \phi(x) + \psi(x)$ and $u_t\ (x,0) = \frac{\phi'(x)\partial(x-ct)}{\partial t} + \frac{\psi'(x)\partial(x+ct)}{\partial t} = -c\phi'(x) + c\psi'(x) = g(x)$, then integrating g from a to x and dividing by c gives: $G(x) = \phi(x) - \psi(x) = (1/c)\int_a^x g(s)\,ds + C$. Hence, ϕ and ψ are found as $(1/2)\,f(x) \pm (1/2c)\int_a^x g(s)ds \pm (c/2)$ respectively. Therefore, solving $u(x,t)$ and $G(x)$ for ϕ and ψ yields:

$$u(x,t) = (1/2)[f(x+ct) + f(x-ct)] + (1/2c)\left[\int_a^{x+ct} g(s)ds - \int_a^{x-ct} g(s)ds\right]$$

which is the d'Alembert's solution given in (2).

Since both PDE and boundary conditions in (1) are linear and homogenous, the method of separation of variables (Asmar, 2004) can also be applied to verify this result. We start with expressing the solution of u by $u(x,t) = \phi(x).T(t)$ to arrive at:$(1/c^2T(t)).\,d^2T/dt^2 = (1/\phi(x)).\,d^2\phi/dx^2 = -\lambda$ which results into two ODEs of the form; $d^2T/dt^2 = -\lambda c^2 T(t)$ and $d^2\phi/dx^2 = -\lambda\phi(x)$. There will be two families of product solutions; the principle of superposition implies that we should be able to solve the initial value problem by considering a linear combination of all product solutions in $u\,(x,t) = \sum_{n=1}^{\infty} A_n\, sin(n\pi/L)x\,.cos(n\pi c/L)t$*+Bnsin(nπ/L)x.sin (nπc/L)t*. The initial conditions are satisfied if: $f(x) = \sum_{n=1}^{\infty} A_n\, sin\left(\frac{n\pi}{L}\right)x$ and $g(x) = \sum_{n=1}^{\infty} B_n (n\pi c/L)\, sin(n\pi/L)x$. From Fourier sine series, we know that $sin(n\pi/L)x$ forms an orthogonal set, therefore: $A_n = 2/L\int_0^L f(x)\,sin(n\pi/L)xdx$ and *Bn=2/nπc0Lgxsin(nπ/L)xdx*. Using the sum-to-product of "*sine*" and "*cosine*", $u(x,t)$can be written as:

$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{2}A_n[sin(n\pi/L)(x+ct) +$ *sin(nπ/L)x−ct+n=1∞Bnsin(nπ/L)x.sin (nπc /L)t*

Substituting x with $(x+ct)$ and $(x-ct)$ in $f(x) = \sum_{n=1}^{\infty} A_n\, sin(n\pi/L)x$, we get: $\frac{1}{2}f(x \pm ct) = \sum_{n=1}^{\infty} A_n\,[sin(n\pi/L)(x \pm ct)]$. Integrating $0\ g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} sin(n\pi/L)x$ from $(x-ct)$ to $(x+ct)$ and then using product-to-sum of "*sine*" and "*cosine*", yields $\sum_{n=1}^{\infty} B_n\ sin\ (n\pi/L)\ x\,.\ sin\ (n\pi c/L)\ t = (1/2c)\int_{x-ct}^{x+ct} g(s)ds$, which combining with (1/*2cfx±ct* will result into d' Alembert's solution.

Heat distribution model: We consider $u = u(x,t)$ representing the temperature in a homogenous rod of length L with perfectly insulated lateral surface. We assume that the ends of the rod at $x = 0$ and $x = L$ are held temperature zero and the initial temperature distribution is a given function, $f = f(x)$. Then u solves the following Initial Boundary Value Problem (IBVP):

$$cu_t = ku_{xx} \ : \ t > 0, 0 < x < L, \quad u(0,t) = u(L,t) = 0 \ : t > 0 \ , \ u(x,0) = f(x), 0 < x < L \tag{3}$$

The solution of this IBVP can be written as a Fourier series. We expect that Eq. (3) completely and unambiguously specify the temperature in the rod. Once we have found a function $u(x,t)$ that meets all three of these conditions, we can be assured that u is the temperature. Using the classical technique for solving the IBVP for the heat equation, the method of separation of variables (Asmar, 2004) allows us to replace the partial derivatives by ordinary derivatives. The idea is to think of a solution $u(x,t)$ as being an infinite linear combination of simple component function, $u_n(x,t), n = 0,1,2, \ldots$, which also satisfy the equation and certain boundary conditions. The solution to the IVBP in (3) then takes the following form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{k}{c}\left(\frac{n\pi}{L}\right)^2 t} sin\frac{n\pi}{L}x \ , \ c_n = \frac{2}{L}\int_0^L f(x). sin\frac{n\pi}{L}xdx, \ \ n = 1,2, \ldots \tag{4}$$

This solution is claimed to be unique. From the theorem of uniqueness of solution, the IVBP in (3) has at most one continuously differentiable solution. This can be proved by assuming $u(x,t)$ and $\vartheta(x,t)$ continuously differentiable functions that satisfy this initial-boundary value problem. If we let $\mathcal{W} = u - \vartheta$, it is also continuously differentiable solution to the boundary value problem in (3). By the maximum principle, $\mathcal{W}$ must attain its maximum at t = 0 and since $\mathcal{W}(x,0) = u(x,0) - \vartheta(x,0) = f(x) - f(x) = 0$. We have $\mathcal{W}(x,t) \leq 0$. Hence $u(x,t) \leq \vartheta(x,t)$ for all $0 \leq x \leq L, \ t \geq 0$. A similar argument using $\widehat{\mathcal{W}} = u - \vartheta$ yields: $\vartheta(x,t) \leq u(x,t)$. Therefore we have $u(x,t) = \vartheta(x,t)$ for all $0 \leq x \leq L, \ t \geq 0$. Thus, there is at most one continuously differentiable solution to the IVBP in (3).

RESULTS AND DISCUSSION

Behavior of wave propagation: The solution of the standing wave in (1) has two parts. The first part of the solution consists of sinusoidal function $sin\,(n\pi x/L)$ multiplied by a time-varying amplitude. The second term of the solution is also sinusoidal function multiplied by sinusoidal time-varying amplitude which will present a node in $x = L/2$ that never moves. For the n^{th} term, we will have $(n-1)$ nodes. In order to describe this behavior, we study the motion of a string with $c = 1$ and initial values $g = 0$ and $\{f(x) = 1 - |x|\!: \ |x| \ \leq 1 \ , \ f(x) = 0 : \ \text{otherwise}\}$. Maple program version 13 (Waterloo Maple Inc®, Ontario) in Appendix 1 was used for computer simulation and to plot the snapshots of the string as provided in Fig. 1. The plots show a wave at its maximum magnitude equal to 1 at $t_0 = 0$. The magnitude of wave then reduces and the wave forms two traveling wave srepelling each other with speed $2c$ and magnitude 0.5. It can also be seen that the magnitude of the two waves remains constant while moving and then doubles as they meet each other. In the other words, the energy of the wave preserves through the time and the motion continues forever. From the solution, it can be observed why the waves are traveling to the right and left. This is because of the replacement of x with $x + ct$ and $x - ct$ which will shift the function to the right and left.We repeat previous experiment with $f = 0$ and $\{g(x) = 1 - |x| \ : \ |x| \leq 1, \ g(x) = 0 : \ otherwise\}$ (Maple® codes in Appendix 2). Plots are provided in Fig. 2 and show that the initial magnitude of the wave is zero, but wave starts growing by the speed of $c = 1 - |x|$ up to the maximum value of $c = 1$. After that point, the wave starts to travel and become wider and wider.

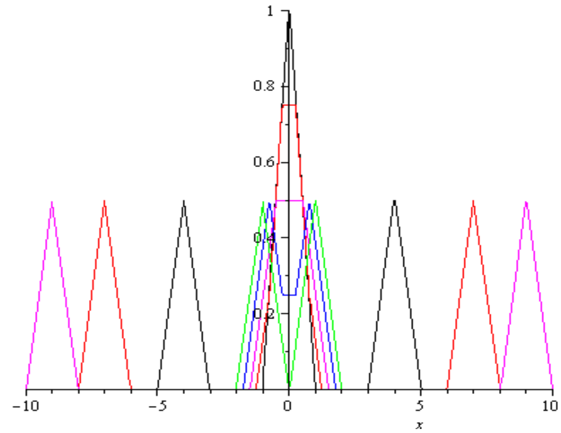

Fig. 1: Plot snapshots of the wave propagation in Eq. (1) with initial conditions, $g = 0$ and $\{f(x) = 1 - |x|: |x| \leq 1\,, f(x) = 0 : otherwise\}$ Snapshots at $t = \{0, 0.25, 0.5, 0.75, 1, 4, 7, 9\}$

When the boundary conditions in wave equations are not given, as in problem 1, the solution will give the infinite string problem consisting travelling waves. Wave equation travels at a finite speed of c to the left and right of the one space dimension (the x axis) as time elapses. The constant term in the wave equation is always positive (squared). Looking at d'Alembert's

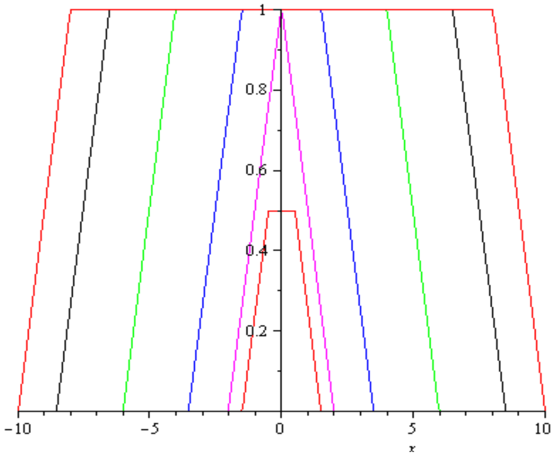

Fig. 2: Plot snapshots of the wave propagation in Eq. (1) with initial conditions, $f = 0$ and $\{g(x) = 1 - |x| : |x| \leq 1, g(x) = 0 : otherwise\}$ Snapshots at $t = \{0, 0.25, 0.5, 0.75, 1, 4, 7, 9\}$

formula in (2), if we let $\ell(x)$ be a function on $\mathbb{R}$, then the function $\ell(x + c)$ where c is a positive constant is the same shape as $\ell(x)$ with its position shifted to the left by c. As an example, plot of $\ell(x) = \sin(x)$, versus plot of $\ell(x + c) = \sin(x + c)$ is given in Fig. 3. These results show that singularities are preserved in wave propagation and are transported along the characteristics with speed of c.

In wave equation, the parameter t is the time. Therefore the function $\ell(x + ct)$ will represent sets of functions that have same shape, but shifted more and more to the left as time passes. The function $\ell(x + ct)$ is called a travelling wave moving to the left with speed c. In a similar manner, the function $\ell(x - ct)$ is also a travelling wave moving to the right with speed c. The integral term in d'Alembert's formula comes from the initial velocity. In the special case when the initial velocity is zero, (that is $u_{tt}(x, 0) = g(x) = 0$), Eq. (2) will be written as: $u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$, which is the sum of travelling waves. The waves are initially superimposed. As time elapses, the two waves repel each other and travel with speed of $2c$, (Each with speed c). More plots of wave equation with different boundary values are shown in Fig. 4 to 7. An observation from the snapshots of plots in Fig. 4 and 5 are waves of half the height that travels in opposite directions with a constant speed and without changing the shape, propagating forever as $t \rightarrow \infty$.

Behavior of heat distribution: To describe the behavior of the IBVP in (3), we calculate the Fourier

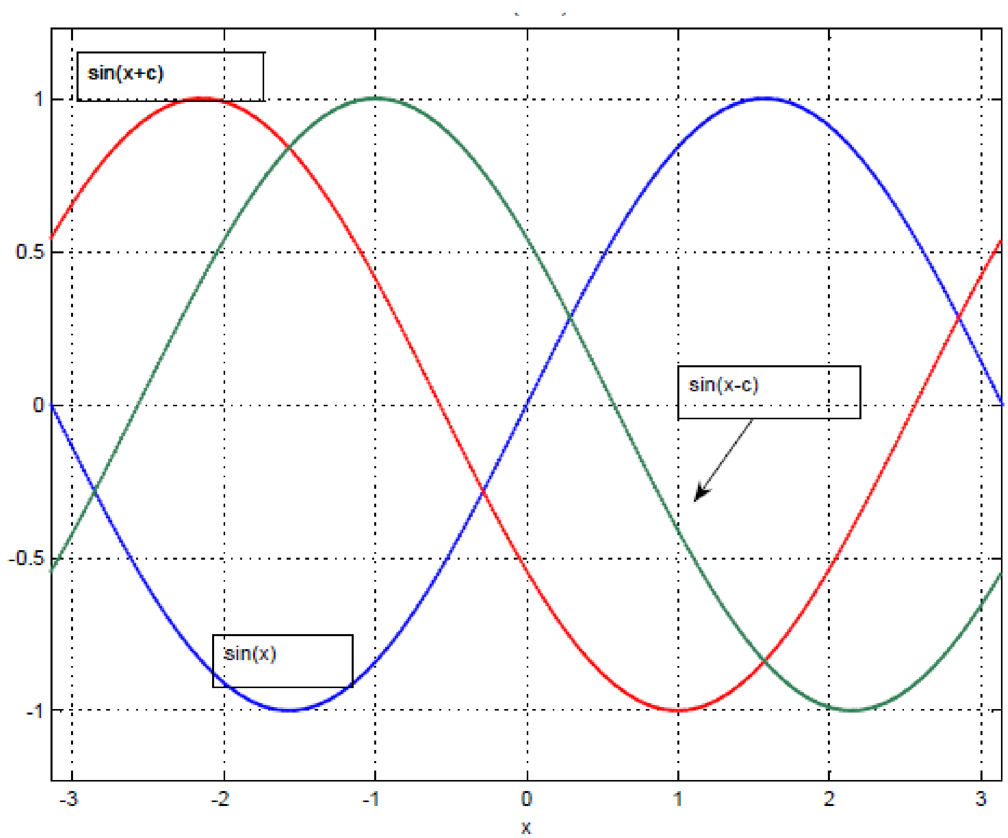

Fig. 3: Example of travelling waves to right and left, shifting **sin** (**x**), **sin**(**x** + **h**) and **sin**(**x** − **h**)

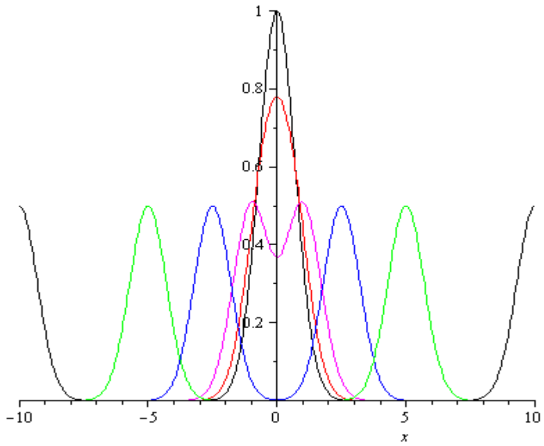

Fig. 4: Infinite string; snapshots at $t = \{0, 0.5, 1, 2.5, 5, 10\}$

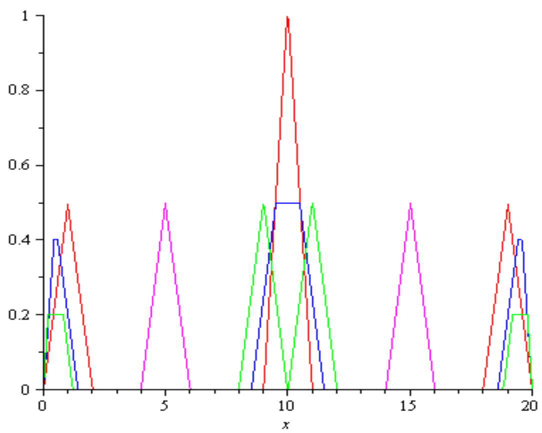

Fig. 5: Finite string (Dirichlet) snapshot at t = {0, 0, .5, 1, 5, 9, 9.6, 9.8}

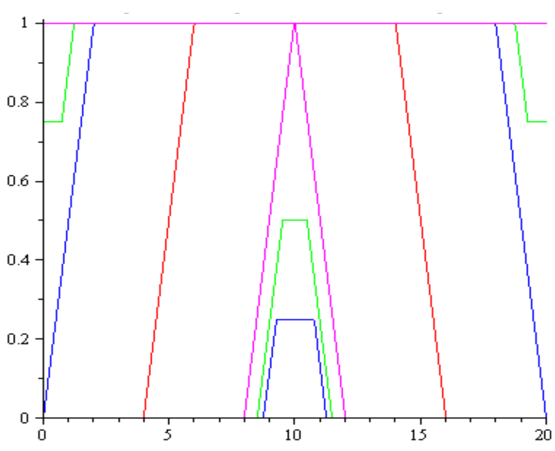

Fig. 6: Finite string (Neumann); snapshots at $t = \{0,\ 0.25,\ 0.5,\ 1,\ 5,\ 9,\ 9.75,\ 10\}$

coefficients explicitly for the special case $f = 1$, which yields: $c_n = (2/n\pi)[1 - cos(n\pi)]$ and the following solution:

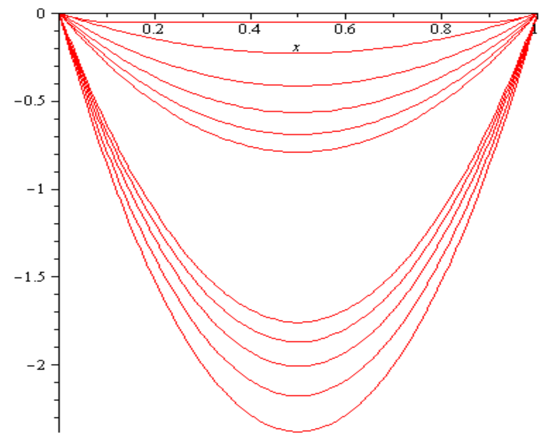

Fig. 7: Wave equation with damping and external force on interval $(0, L)$ snapshots at $t = \{0.1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-cos(n\pi)}{2n}e^{-\frac{k}{c}(n^2\pi^2)t}sin(n\pi x) \quad (5)$$

Results were then plotted as provided in Fig. 8 for $\beta = k/c = 1$ and $n = 50$. Based on this solution and from physical interpretation of the problem, the boundary conditions state that the temperature, regardless of the change in time, at the two ends of the rod is zero. It means that at initial condition which requires $f = 1$, we have zero temperature at the endings. This is a very ideal theoretical assumption that requires perfect isolation of the material at those points. According to the initial condition, the temperature at $t_0 = 0$ should be equal to 1 for any $x \in (0, L)$. In the other words, the temperature should be zero at $x = 0$ and suddenly becomes 1 at $x = \varepsilon$ or $x = L - \varepsilon$. This step change will always result overshoots and corresponding undershoots. Mathematically called Gibbs phenomenon, this situation occurs only when a finite series of Eigen functions approximates a discontinuous functions. In general, there is an overshoot and undershoot of approximately 9% of the jump discontinuity (Hazewinkel, 2001). It can also be observed from plots of $u(x,t)$ in Fig. 8 that as $t \to \infty$, $u(x,t)$ starts decreasing rapidly with time. Since the temperature at the boundaries is zero, it is expected that the temperature of the rod at any point $x \in (0, L)$ becomes equal to the temperature at the boundaries as $t \to \infty$. In the other words, the temperature at no point of the rod in Eq. (3) will ever become greater than the initial value, therefore the energy is not conserved. This is not in contrast with the rule of conservation of energy since the energy from the hot rod is transferred to its soundings, so depending on the boundaries of our system, the total energy is still constant. From the plots of $u(x,t)$ shown in Fig. 9 and 10 at time $t = 0.001$ and $t = 0.1$, it can be seen that

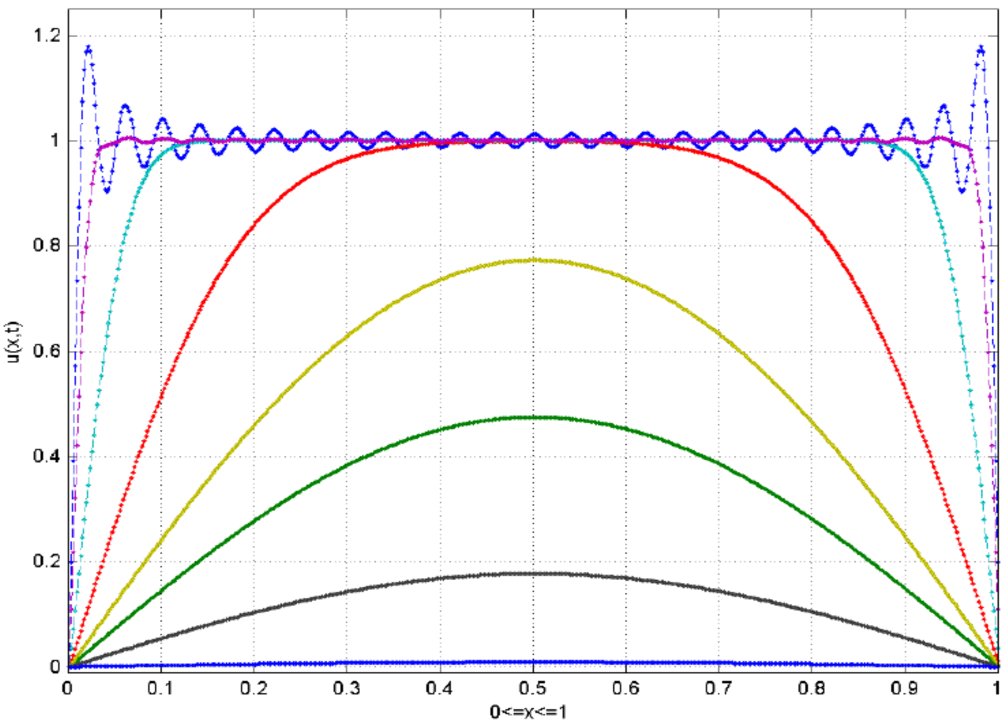

Fig. 8: Solution to the heat equation in (3) with $f = 1,\ \beta = 1$; snapshot at $t = \{0,0.1,0.5,2.5,10\}$

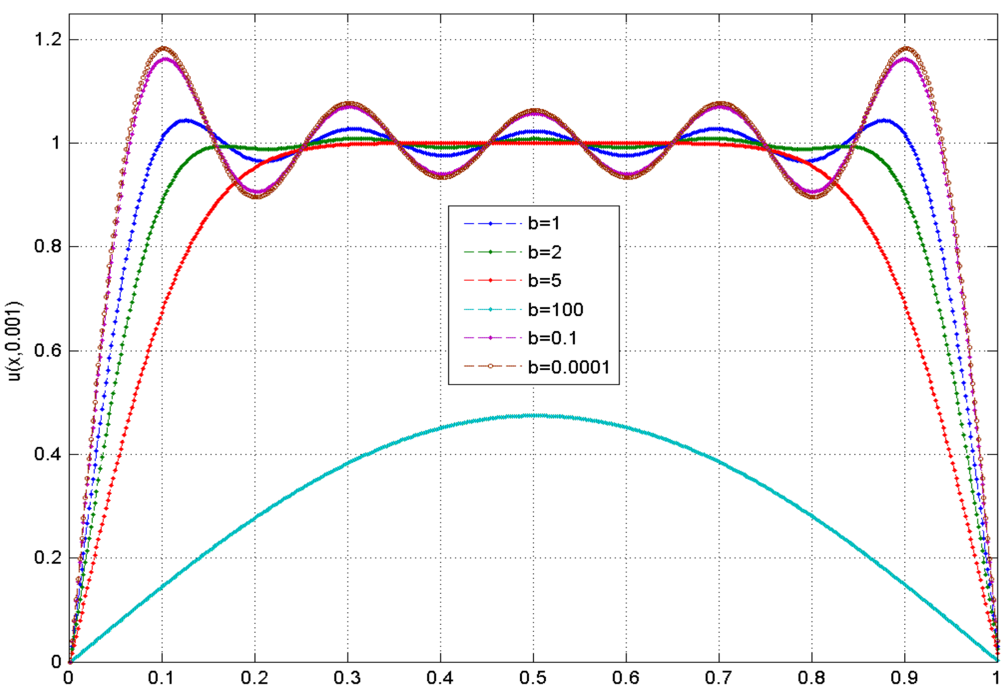

Fig. 9: Solution to the heat equation in (3) with $f = 1$, at $t = 0.001$ and $\beta = \{0.0001,\ 0.1,\ 1,\ 2,\ 5,\ 100\}$

$u(x,t)$ decreases with smaller values of β, however the behavior of the solution is still similar to the case in Fig. 8 and $u(x,t)$ starts decreasing with time.

If we change the initial condition from a constant value to a function term, for example, $f(x) = exp(-x^2)$, the solution can be written as follow (Asmar, 2004):

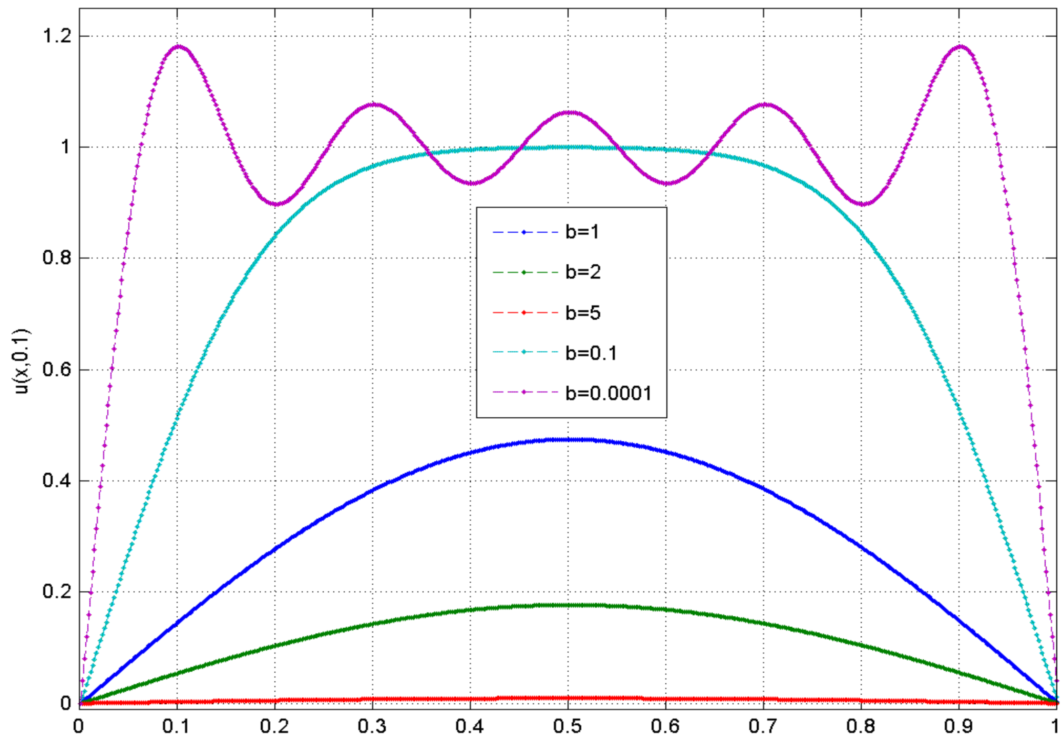

Fig. 10: Solution to the heat equation in (3) with $f = 1$, at $t = 0.1$ and $\beta = \{0.0001, 0.1, 1, 2, 5\}$

$$u(t,x) = \int_{-\infty}^{\infty} G_k(t, x-y)\, f(y)dy,\ G_k(t,x) = \frac{exp\,(-\frac{x^2}{4\beta t})}{\sqrt{4\pi\beta t}} \quad (6)$$

Using the Maple code in Appendix 3, snapshots of the solution in (6) was plotted in Fig. 11 where similar behavior like the previous case is observed. The reason for $u(x,t)$ to start decaying over timeis because $G_k(t, x-y) \to 0$ as $t \to \infty$. This result can be confirmed analytically by letting $\zeta = x - y$ and $\rho = \zeta/\sqrt{\beta t}$ in (6), hence the solution takes the following form given in (7) which will be bounded if the initial function f is integrable. Similar experiment with a non-integrable function like $f(x) = exp(x)$ shows that the solution travels from right to left and never decays:

$$u(t,x) = \int_{-\infty}^{\infty} \frac{exp\,(-\rho^2/4)}{\sqrt{4\pi}} f\left(x - \rho\sqrt{\beta t}\right)d\rho \quad (7)$$

To further experiment with the heat distribution, we study the backward heat equation, $u_t = -\beta u_{xx}$, subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$. Using the general solution in (4), If we let $\beta = -1$, $L = 1$, with initial condition$u(x,0) = f(x) = 1$, the solution will then take the following form:

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-cos(n\pi)}{2n} e^{(n^2\pi^2)t} sin(n\pi x) \quad (8)$$

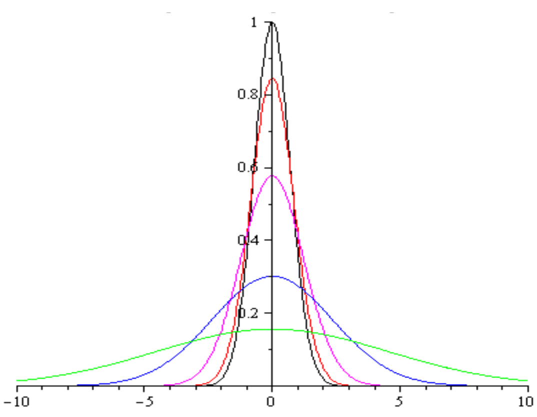

Fig. 11: Solution to heat equation in (3) with $f(x) = e^{-x^2}$, $\beta = 1$ snapshot at $t = \{0, 0.1, 0.5, 2.5, 10\}$

For the same $\beta = -1$ and $L = 1$, if we let $u(x,0) = f(x) = 1 + \frac{1}{n} sin\frac{n\pi x}{L}$, the coefficient c_n can be calculated as: $c_n = \frac{2}{n\pi}(1 - cosn\pi) + \frac{1}{n}, (n = 1, 2, \ldots)$ and the solution will be given by:

$$u(x,t) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}(2(1 - cosn\pi) + \pi)\, e^{(n^2\pi^2)t} sin(n\pi x) \quad (9)$$

Both solutions in (8) and (9) were implemented in Delphi programming language (Borland®, Austin, TX)

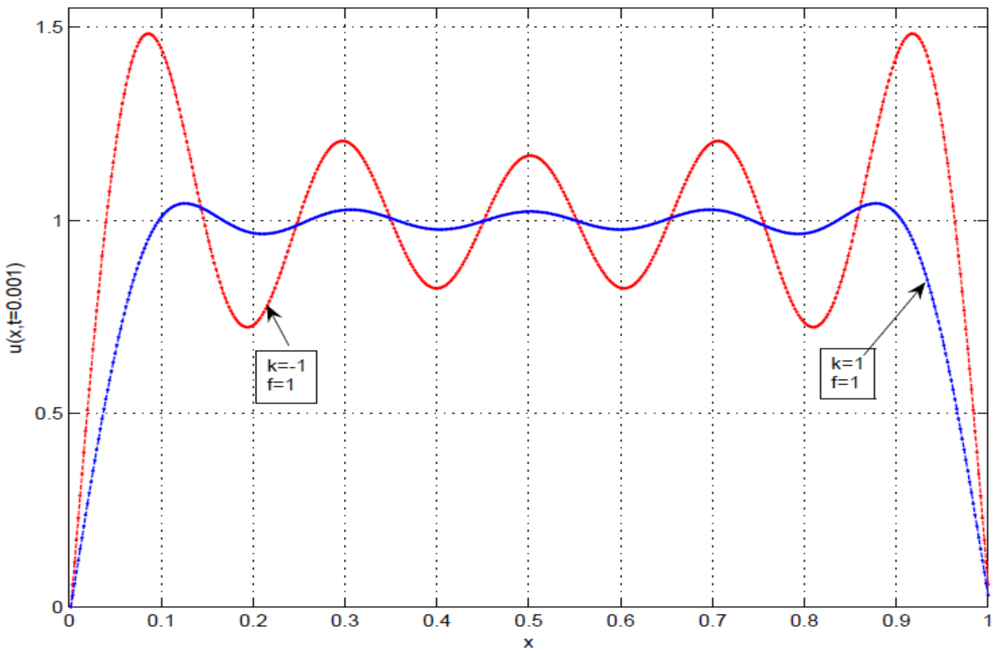

Fig. 12: Comparing Eq. (5) with Eq. (8) at $t = 0.001$

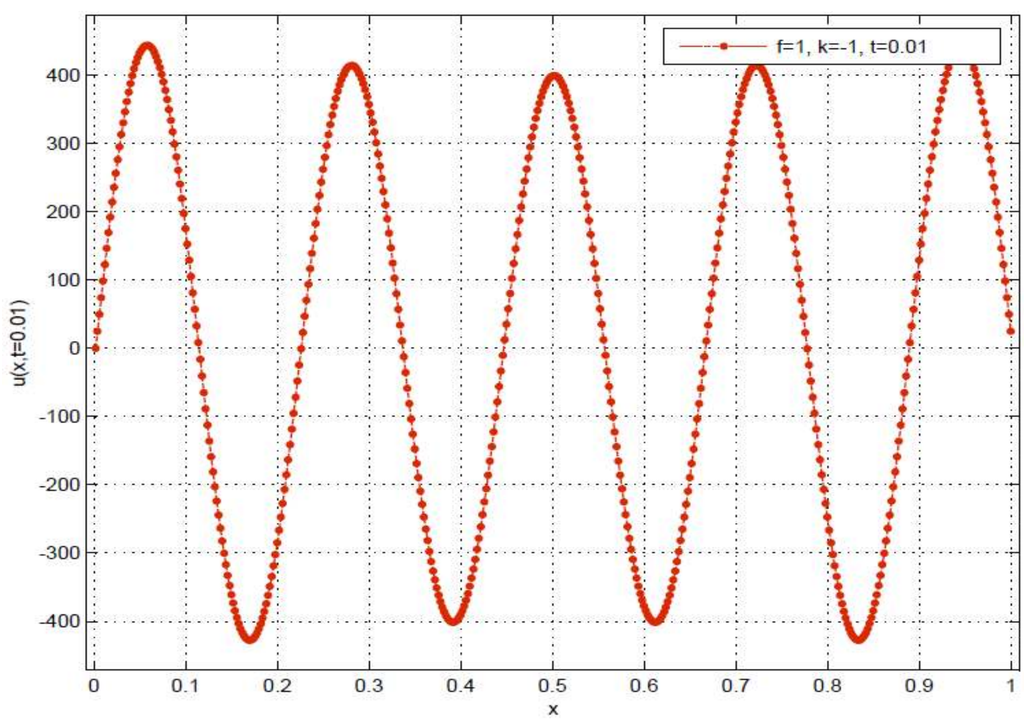

Fig. 13: Plot of equation (8) at $t = 0.01$; (the response is highly oscillatory and unstable)

environment (codes are provided in Appendix 4 and 5) to generate the plots shown in Fig. 12 to 15. It can be seen that if the initial data are changed by an arbitrary small amount, for example, if $f(x) \to f(x) + \frac{1}{n}\sin(n\pi x/L)$, for large n, then solution changes by a large amount. It is therefore concluded that backward heat equation is not a well-posed problem. This can be explained by comparing plots of Eq. (5) and (8) as shown in Fig. 12 and 13 respectively. We observe that for the same initial condition, the magnitude of Eq. (8) is larger and grow unbounded in finite time, which is due to its positive exponential term ($\lim_{n\to\infty} e^{n^2\pi^2 t} = \infty$) that makes the system to oscillate

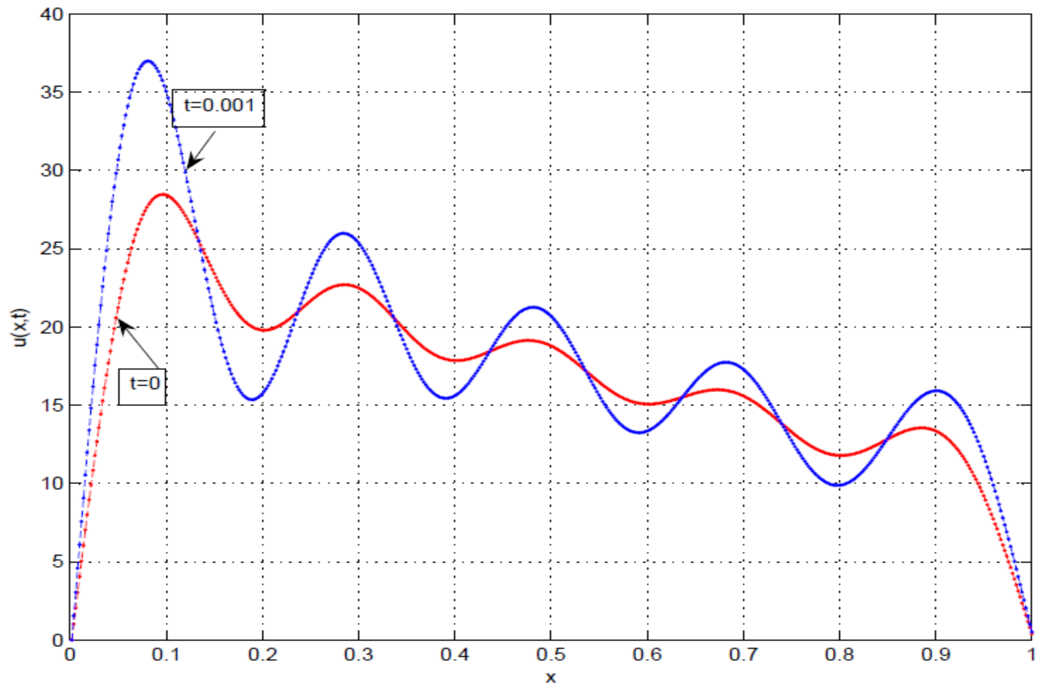

Fig. 14: Plot of Eq. (9) at $t = 0$ and $t = 0.001$; (big change in $u(x, t)$ compared with Fig. 12)

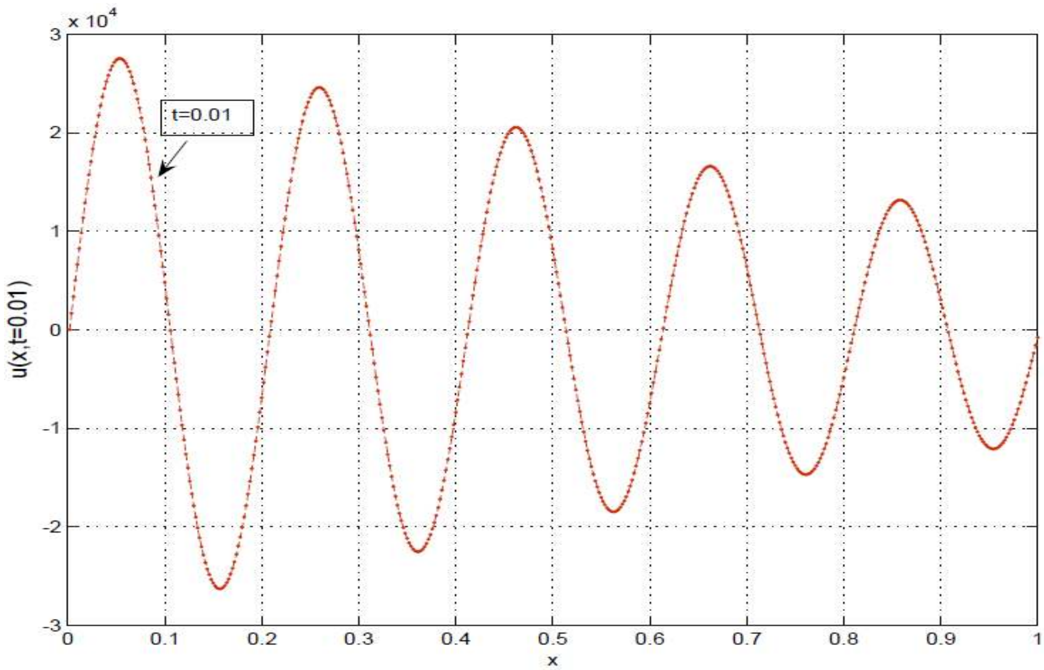

Fig. 15: Plot of Eq. (9) at $t = 0.01$; (big change in $u(x, t)$ compared with Fig. 13)

and become unstable for small change in the input. For large n, $f(x) \rightarrow f(x) + \frac{1}{n} sin \frac{n\pi x}{L}$ is equal to $f(x) \rightarrow f(x) + \varepsilon$, (since $n > 0$, $-1 \leq sin \frac{n\pi x}{L} \leq 1$, thus $-1 < \frac{1}{n} sin \frac{n\pi x}{L} < 1$). It can be seen from Fig. 14 that the magnitude of solution for the same time snap is almost 37 for $f(x) = 1 + \varepsilon$ while the corresponding magnitude for $f(x) = 1$ is almost 1.5. This can also be seen for larger $t's$. For t = 0.01, we see a big change in the magnitude of solution, from 450 in Fig. 13 to 27000 in Fig. 15.

One more interesting observation from the solution to heat equation given in (4) is that all the terms in (4) are always nonnegative (because n is a positive integer,

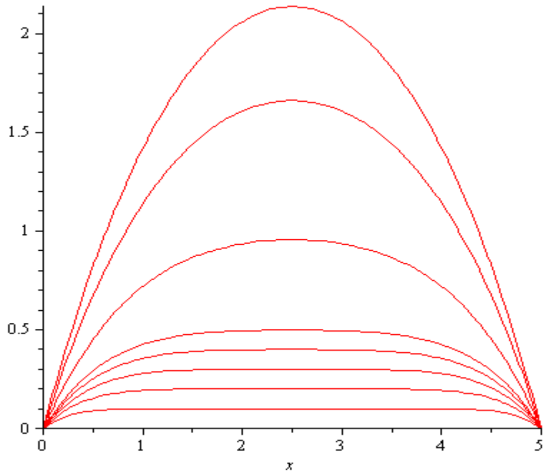

Fig. 16: Heat equation with source on interval $(0, L)$; snapshots at $t = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 1,\ 2,\ 3\}$

$0 \le x/L \le 1$, $sin\frac{n\pi}{L}x \ge 0$, $exp(-k/c(n\pi/L)^2 t) \ge 0$, $lim_{t\to\infty}\exp(-k/c(n\pi/L)^2 t) = 0$, $c_n \ge 0$). Therefore, the solution to heat equation, $u(x,t)$ cannot be negative unless the initial condition is negative. This property is also obvious from snapshots plot of heat equation as shown in Fig. 16. The graphs of heat solution also show that information is gradually lost in heat distribution, in the other words, heat from higher temperatures is dissipated to a lower temperature, but it will not be clear what the original temperatures were after some elapses of time.

CONCLUSION

Comparison between wave propagation as governed by the wave equation and heat distribution, as governed by the heat equation were discussed in this paper. Issues such as finite vs. infinite speed propagation, propagation of singularities vs. instantaneous smoothing, conservation of wave profile vs. averaging and time reversibility vs. irreversibility were addressed. Arguments were supported with examples and graphical evidences.

A model for the motion of a vibrating string was provided as an example of wave equation. It was shown that the solution has two parts. The boundary conditions dictate the condition of the string at the ending points, (whether it is held fixed or not). The initial conditions then specify the initial displacement and the initial velocity of each point. An important observation from the snapshots plots of wave propagation is that wave travels with a constant speed and continue propagating forever as time goes to infinity. Depending on the physic of the problem, wave equation transfers a form of energy, like a vibrating string of a guitar which transfers sound energy or electromagnetic waves which transfer light. According to the rule of conservation of energy, the total energy of a system is constant, thus theoretically wave should keep travelling forever. In the other words, the solutions of the wave equation does not decay as $t \to \infty$. For the heat equation, although it also transfer energy, but there is energy dissipation built into the high order terms, however there are special nonlinear forms of heat distribution (radiation) that can have travelling wave solution. In summary, the heat distribution is a non-reversible equation, it does not conserve energy, cannot exceed the value at initial condition or become negative if the initial condition is not negative The distribution speed in heat equation is very fast, uniformly distributed and dies down as time goes to infinity. It was also shown that heat equation follows maximum principle. In the other side, the propagation speed in wave equation is finite. It is a reversible equation, can conserve energy and can exceed the value of initial condition (when two waves coincide) and bounce up and down and distribute forever. Wave equation does not follow the maximum principle and does not depend on initial conditions to have negative value.

Appendix 1: Maple program used to solve the wave problem with $f(x) = 1 - |x|$ if $|x| \le 1$, $f(x) = 0$ otherwise and g = 0

```
restart: with (plots):
(1/c^2) *u [tt] = u [xx];
u (x, 0) = f (x); u [t] (x, 0) = g (x);
u (x, t) = (f (x-c*t) + f (x + c*t)) /2 + (1/ (2*c)) *int (g (y), y = x -
c*t..x + c*t);
c: = 1; f (x): = piecewise (abs (x) < = 1, 1-abs (x), 0); g (x): = 0;
`Computing u (x, t)...`;
f (x): = simplify (f (x)): g (x): = simplify (g (x)): G (x): = int (g (x),
x):
f0: = unapply (f (x), x): G0: = unapply (G (x), x): u: = (x, t) - (f0 (x +
c*t) + f0 (x - c*t)) /2 + (G0 (x + c*t) - G0 (x - c*t)) / (2*c):
`Finished.`;
t0: = 0:
plot (u (x, t0), x = -5..5, 0..1, numpoints = 250,
title = cat (`Snapshot at`, convert ('t' = t0, string)));
Sequence of snapshots at times [t1, t2, t3, ..., tn]:
tseq: = [0, 0.25, 0.5, 0.75, 1, 4, 7, 9]:
plot ([seq (u (x, t), t = tseq)], x = -10..10, 0..1, numpoints = 250,
color = [black, red, magenta, blue, green],
title = cat (`Snapshots at `, convert ('t' = tseq, string)));
Create movie (t = t0 .. tn, n + 1 frames):
t0: = 0: tn: = 12: n: = 60:
h: = (tn - t0) /n: `Generating graphs... `;
snaps: = seq (plot (u (x, t0 + i*h), x = -10..10, 0..1, numpoints = 250),
i = 0..n): `Finished.`;
display (snaps [1..n + 1], insequence = true);
```

Appendix 2: Maple program used to solve the wave problem with $f = 0$ and $g(x) = 1 - |x|\ if\ |x| \le 1$ and $g(x) = 0$ otherwise

```
restart: with (plots):
(1/c^2) *u [tt] = u [xx];
u (x, 0) = f (x); u [t] (x, 0) = g (x);
u (x, t) = (f (x - c*t) + f (x + c*t)) /2 + (1/ (2*c))* int (g (y), y = x -
c*t..x + c*t);
c: = 1; f (x): = 0; g (x): = piecewise (abs (x)< = 1, 1, 0);
`Computing u (x, t)...`;
f (x): = simplify (f (x)): g (x): = simplify (g (x)): G (x): = int (g (x),
x):
f0: = unapply (f (x), x): G0: = unapply (G (x), x): u: = (x, t) - (f0 (x +
c*t) + f0 (x - c*t)) /2 + (G0 (x + c*t) - G0 (x - c*t)) /(2*c):
`Finished.`;
```

t0: = 0:
plot (u (x, t0), x = -5..5, 0..1, numpoints = 250,
title = cat (`Snapshot at `, convert ('t' = t0, string)));
Sequence of snapshots at times [t1, t2, t3, ..., tn]:
tseq: = [0, 0.5, 1, 2.5, 5, 7.5, 9]:
plot ([seq (u (x, t), t = tseq)], x = -10..10, 0..1, numpoints = 250,
color = [black, red, magenta, blue, green],
title = cat (`Snapshots at `, convert ('t' = tseq, string)));
Create movie (t = t0 .. tn, n + 1 frames):
t0: = 0: tn: = 12: n: = 60:
h: = (tn - t0) /n: `Generating graphs...`;
snaps: = seq (plot (u (x, t0 + i*h), x = -10..10, 0..1, numpoints = 250), i = 0..n): `Finished.`;
display (snaps [1..n + 1], insequence = true);

Appendix 3: Maple program used to solve heat equation with $f(x) = e^{-x2}$

```
restart: with (plots):
u [t] = k*u [xx];
u (0, x) = f (x);
u (t, x) = int (G [k] (x - y) *f (y), y = -infinity..infinity);
G [k] (t, x) = exp (-x^2/ (4*k*t)) /'sqrt (4*Pi*k*t)';
k: = 1; f (x): = exp (-x^2);
`Computing u (t, x)...`;
f (x): = simplify (f (x)): f0: = unapply (f (x), x): fudge: = t - piecewise (t = 0, 1, t):
G00: = (t, x) - exp (-x^2/ (4*k*t)) /sqrt (4*Pi*k*t): G0: = (t, x) - G00 (fudge (t), x):
u: = (t, x) - piecewise (t0, int (G0 (t, z) *f0 (x - z), z = -infinity.. infinity), f0 (x)):
`Finished.`;
t0: = 0.2:
plot (u (t0, x), x = -10..10, 0..1, numpoints = 250,
title = cat (`Snapshot at `, convert ('t' = t0, string)));
tseq: = [0, 0.1, 0.5, 2.5, 10]:
plot ([seq (u (t, x), t = tseq)], x = -10..10, 0..1, numpoints = 250,
color = [black, red, magenta, blue, green],
title = cat(`Snapshots at `, convert ('t' = tseq, string)));
t0: = 0: tn: = 10: n: = 50:
h: = (tn - t0) /n: `Generating graphs…`;
snaps: = seq (plot (u (t0 + i*h, x), x = -10..10, 0..1, numpoints = 250), i = 0..n): `Finished.`;
display (snaps [1..n + 1], insequence = true);
```

Appendix 4: Delphi code for backward heat Eq. (8), with $k/c = 1$ and $\Delta x = 0.002$ and $n = 50$

```
procedure TForm1.Button1Click (Sender: TObject);
varL, beta, cn, en, t, sin_n, x, fx:real;
n, x_inc:integer;
label start;
begin
memo1.Clear;
//Define L and beta here
L: = 1;  beta: = strtofloat (edit2.Text);
//Define time snapshot here
t: = strtofloat (edit1.Text);
start:
fx: = 0;
for n: = 1 to spinedit1.Value do
begin
fx: = fx + ((2/ (n*pi))* (1-cos (n*pi))* (exp (- (beta*n*n*pi*pi*t/ (L*L)))) * (sin (n*pi*x/L)));
end;
memo1. Lines. Add (floattostr (fx));
x: = x + (L/500);
if x< = L then goto start;
end;
end.
```

Appendix 5: Delphi code for backward heat Eq. (9), with $k/c = 1$ and $\Delta x = 0.002$ and $n = 50$

```
procedure TForm1.Button1Click (Sender: TObject);
varL, beta, cn, en, t, sin_n, x, fx:real;
n, x_inc:integer;
label start;
begin
memo1.Clear;
//Define L and beta here
L: = 1; beta: = strtofloat (edit2.Text); //beta = -1 for this problem
t: = strtofloat (edit1.Text);
start:
fx: = 0;
for n: = 1 to spinedit1.Value do
begin
fx: = fx + (1/n*pi) * (2* (1-cos (n*pi)) + pi) *exp (- (beta*n*n*pi*pi*t/ (L*L))) *sin (n*pi*x/L);
end;
memo1.Lines.Add (floattostr (fx));
x: = x + (L/500);
if x< = L then goto start;
end;
end.
```

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