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Research Article

Use of Homotopy Perturbation Method for Solving Multi-point Boundary Value Problems

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Abstract: Homotopy perturbation method is used for solving the multi-point boundary value problems. The approximate solution is found in the form of a rapidly convergent series. Several numerical examples have been considered to illustrate the efficiency and implementation of the method and the results are compared with the other methods in the literature.

Keywords: Approximate solution, homotopy perturbation method, linear and nonlinear problems, multi-point boundary value problems

INTRODUCTION

Multipoint boundary value problems arise in applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be given as a multi-point boundary value problem (Moshiinsky, 1950). Hajji (2009), considered the multi-point boundary value problems which occurs in many areas of engineering applications such as in modelling the flow of fluid such as water, oil and gas through ground layers, where each layer constitutes a sub domain. In Timoshenko (1961), many problems in the theory of elastic stability are handled by multi-point problems. In Geng and Cui (2010) large size bridges are sometimes contrived with multi-point supports which correspond to a multi-point boundary value condition. Many authors studied the existence and multiplicity of solutions of multi-point boundary value problems (Eloe and Henderson, 2007; Feng and Webb, 1997; Graef and Webb, 2009; Henderson and Kunkel, 2008; Liu, 2003). Some research works are available on numerical analysis of the multi-point boundary value problems. Numerical solutions of multi-point boundary value problems have been studies by Geng (2009), Lin and Lin (2010), Tatari and Dehghan (2006) and Wu and Li (2011). Siddiqi and Akram (2006a, b) presented the solutions of fifth and sixth order boundary value problems using non-polynomial spline technique. In (Siddiqi *et al*., 2012a, b) and (Siddiqi and Iftikhar, 2013a) solutions of seventh order boundary value problems are discussed. Recently, Akram and Rehman (2013a) used the reproducing Kernel space method to solve the eighth-order boundary value problems and in Akram and Rehman (2013b) find the solution of a class of sixth order boundary value problems using the reproducing kernel space method. Siddiqi and Iftikhar (2013b) presented the solution of higher order boundary value problems using the homotopy analysis method.

He (1999, 2003, 2004, 2005) developed the homotopy perturbation method for solving nonlinear initial and boundary value problems by combining the standard homotopy in topology and the perturbation technique. By this method, a rapid convergent series solution can be obtained in most of the cases. Usually, a few terms of the series solution can be used for numerical calculations. Chun and Sakthivel (2010), implement the homotopy perturbation method for solving the linear and nonlinear two-point boundary value problems. The convergence of the homotopy perturbation method was discussed in Biazar and Ghazvini (2009), He (1999), Hussein (2011) and Turkyilmazoglu (2011). This method has been successfully applied to ordinary differential equations, partial differential equations and other fields (Belndez *et al*., 2007; Dehghan and Shakeri, 2008; He, 1999, 2003, 2004, 2005; Rana *et al*., 2007; Yusufoglu, 2007).

In this study, the application of the homotopy perturbation method for finding an approximate solution for multi-point boundary value problems has been investigated.

ANALYSIS OF THE HOMOTOPY PERTURBATION METHOD (HE, 1999)

Consider the nonlinear differential equation:

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$$L(u)+N(u)=f(r),\quad r\in\Omega \tag{1}$$

With boundary conditions:

$$B(u,\frac{\partial u}{\partial n})=0,\quad r\in\Gamma \tag{2}$$

where,

L : A linear operator
N : A nonlinear operator
f(r) : A known analytic function
B : A boundary operator
Γ : The boundary of the domain Ω

By He's homotopy perturbation technique (He, 1999), define a homotopy $v(r,p):\Omega\times[0,1]\to R$ which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0, \tag{3}$$

or:

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0, \tag{4}$$

where, $r\in\Omega$, $p\in[0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (1) which satisfies the boundary conditions. Clearly:

$$H(v,0)=L(v)-L(u_0)=0, \tag{5}$$

$$H(v,1)=L(v)+N(v)-f(r)=0, \tag{6}$$

As p changes from 0 to 1, then v(r,p) changes from $u_0(r)$ to $u(r)$ This is called a deformation and $L(v)-L(u_0)$, $L(v)+N(v)-f(r)$ are said to be homotopic in topology. According to the homotopy perturbation method, firstly, the embedding parameter p can be used as a small parameter and assume that the solution of Eq. (3) and (4) can be expressed as a power series in p, that is:

$$v=v_0+pv_1+p^2v_2+\cdots \tag{7}$$

For p = 1, the approximate solution of Eq. (1) therefore, can be expressed as:

$$v=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots \tag{8}$$

The series in Eq. (8) is convergent in most cases and the convergence rate of the series depends on the nonlinear operator, see (Biazar and Ghazvini, 2009; He, 1999). Moreover, the following judgments are made by He (1999, 2006):

- The second order derivative of N(v) w.r.t. v must be small as the parameter may be reasonably large, i.e., $p\to 1$
- $\left\| L^{-1}\left(\frac{\partial N}{\partial v}\right)\right\|$ must be smaller than one, so that, the series converges

To implement the method, several numerical examples are considered in the following section.

NUMERICAL EXAMPLES

Example 1: Consider the following third-order linear differential equation with three point boundary conditions:

$$\left.\begin{array}{l} u'''(x)-k^2u'(x)+a=0,0\le x\le 1,\\ u'(0)=u'(1)=0,u(0.5)=0.\end{array}\right\} \tag{9}$$

The exact solution of the Example 1 is:

$$u(x)=\frac{a}{k^3}(\sinh\frac{k}{2}-\sinh kx)+\frac{a}{k^2}(x-\frac{1}{2})+\frac{a}{k^3}\tanh\frac{k}{2}(\cosh kx-\cosh\frac{k}{2})$$

where, the constants are k = 5 and a = 1 (Akram *et al*., 2013c; Ali *et al*., 2010; Saadatmandi and Dehghan, 2012; Tirmizi *et al*., 2005).

Using the homotopy perturbation method, the following homotopy for the system (1) is constructed:

$$u'''=p[25u']-1, \tag{10}$$

where, $p\in[0,1]$ is the embedding parameter. Assume that the solution of Problem (1) is:

$$u=u_0+pu_1+p^2u_2+\cdots \tag{11}$$

Substituting Eq. (3) in Eq. (2) and equating the coefficients of like powers of p, gives the following set of differential equations:

$$p^0:u_0'''=-1,\ u_0'(0)=0,u_0(0)=A,u_0''(0)=B$$

$$p^1:u_1'''=25u_0',\ u_1'(0)=0,u_1(0)=0,u_1''(0)=0$$

$$p^2:u_2'''=25u_1',u_2'(0)=0,u_2(0)=0,u_2''(0)=0\ \vdots$$

Table 1: Comparison of numerical results for Example 1

x	Exact solution	Approximate Series solution	Absolute Error Present method	Absolute error (Tirmizi *et al*., 2005)	Absolute error (Ali *et al*., 2010)	Absolute error (Akram *et al*., 2013c)
0.0	-0.01210710	-0.012107100	2.07338E-10	0.00003515	1.298 E-10	8.37E-07
0.1	-0.01126850	-0.011268500	2.02182E-10	0.00003850	3.099E-09	3.39E-07
0.2	-0.00922221	-0.009222210	1.85398E-10	0.00003028	6.959E-09	9.16E-08
0.3	-0.00646687	-0.006466870	1.52702E-10	0.00002231	1.086E-09	7.22E-08
0.4	-0.00332019	-0.003320190	9.57487E-11	0.00001403	1.065E-08	7.86E-08
0.5	0.000000000	-4.03581E-18	4.03581E-18	0.00000700	6.155E-17	6.55E-08
0.6	0.003320190	0.0033201900	1.58981E-10	0.00001260	1.065E-08	6.35E-08
0.7	0.006466870	0.0064668700	4.21657E-10	0.00001260	1.086E-09	6.26E-08
0.8	0.009222210	0.0092222100	8.52404E-10	0.00001956	6.959E-09	9.54E-08
0.9	0.011268500	0.0112685000	1.51972E-09	0.00002741	3.099E-09	3.37E-07
1.0	0.012107100	0.0121071000	2.12120E-09	0.00002395	1.298E-10	8.48E-07

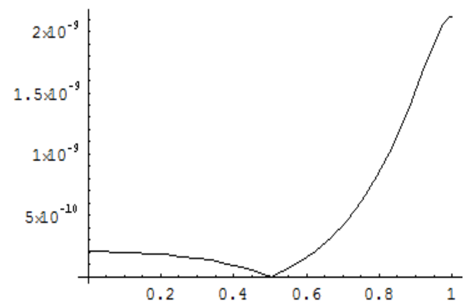

Fig. 1a: Plot of errors

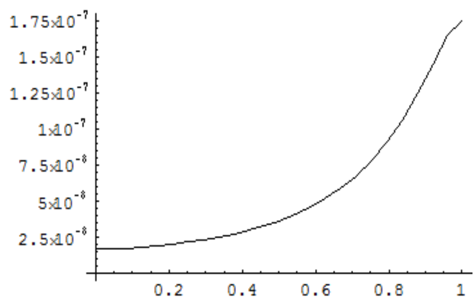

Fig. 1b: Plot of errors

where, A and B are unknown constants to be determined. The corresponding solutions for the above system of equations are the series solution given as:

$$u_0(x) = \frac{1}{6}(6A + 3Bx^2 - x^3)$$

$$u_1(x) = \frac{5}{24}(5Bx^4 - x^5)$$

$$\vdots$$

Using the 11-term approximation, that is:

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_{10}(x) \tag{12}$$

Imposing the boundary conditions of the system (1) on Eq. (12) the values of the constants A and B can be obtained as:

$$A = -0.012107085822126442,\ B = 0.19732286064025403.$$

Then, the series solution can be expressed as:

$U(x) = -0.0121071 + 0.0986614x^2 - 0.16667x^3 + 0.205545x^4 - 0.208333x^5 + 0.171287x^6 - 0.124008x^7 + 0.0764675x^8 - 0.0430583x^9 + 0.021241x^{10} - 0.009785x^{11} + 0.00402291x^{12} + O(x^{13})$

The comparison of the approximate series solution of the problem (1) with the results of methods in Akram *et al*. (2013c), Ali *et al*. (2010), Saadatmandi and Dehghan (2012) and Tirmizi *et al*. (2005) is given in Table 1, which shows that the method is quite efficient. In Fig. 1a and 1b errors $|U - u_{Exact}|$ and $|U - u_{Exact}| / |u_{Exact}|$ are plotted, respectively. Figure 1 shows that the method is in excellent agreement with (Tatari and Dehghan, 2006).

Example 2: Consider the linear fourth-order nonlocal boundary value problem:

$$\left.\begin{array}{l} u^{(4)}(x) - e^x u^{(3)}(x) + u(x) = 1 - e^x\cosh(x) + 2\sinh(x), 0 \le x \le 1 \\ u\left(\frac{1}{4}\right) = 1 + \sinh\left(\frac{1}{4}\right), u^{(1)}\left(\frac{1}{4}\right) = 1 + \cosh\left(\frac{1}{4}\right), \\ u^{(2)}\left(\frac{1}{4}\right) = \sinh\left(\frac{1}{4}\right), u\left(\frac{1}{2}\right) - u\left(\frac{3}{4}\right) = \sinh\left(\frac{1}{2}\right) - \sinh\left(\frac{3}{4}\right). \end{array}\right\}$$

The exact solution of the problem (2) is $u(x) = 1 + \sinh(x)$ (Lin and Lin, 2010; Wu and Li, 2011).

Using the homotopy perturbation method, the following homotopy for the system (5) is constructed:

$$u^{(4)}(x) = 1 - e^x\cosh(x) + 2\sinh(x) + p[e^x u^{(3)}(x) - u(x)] \tag{13}$$

where, $p \in [0,1]$ is the embedding parameter. Assume that the solution of Problem (5) is:

Table 2: Comparison of numerical results for problem (2)

x	Exact solution	Approximate series solution	Absolute error present method	Absolute error in (Lin and Lin, 2010)	Absolute error in (Wu and Li, 2011)
0.0	1.00000	1.00000	1.95677E-09	1.02E-4	2.54E-8
0.1	1.10017	1.10017	5.83738E-10	1.81E-5	4.70E-9
0.2	1.20134	1.20134	1.04897E-10	5.33E-7	1.39E-10
0.3	1.30452	1.30452	9.55858E-11	3.94E-7	1.25E-10
0.4	1.41075	1.41075	3.01129E-10	7.60E-6	2.40E-9
0.5	1.52110	1.52110	2.05711E-09	2.36E-5	7.58E-9
0.6	1.63665	1.63665	5.14444E-09	3.90E-5	1.13E-8
0.7	1.75858	1.75858	6.73259E-09	3.73E-5	4.30E-9
0.8	1.88811	1.88811	1.47844E-08	2.42E-6	2.80E-8
0.9	2.02652	2.02652	1.55269E-07	1.06E-4	1.05E-7
1.0	2.17520	2.17520	7.92993E-07	3.05E-4	2.52E-7

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{14}$$

Substituting Eq. (13) in Eq. (14) and equating the coefficients of like powers of p, gives the following set of differential equations:

$$p^0 : u_0^{(4)}(x) = 1 - e^x \cosh(x) + 2\sinh(x),$$

$$u_0(0) = A, u_0^{(1)}(0) = B, u_0^{(2)}(0) = C, u_0^{(3)}(0) = D,$$

$$p^1 : u_1^{(4)}(x) = e^x u_0^{(3)} - u_0,$$

$$u_1(0) = 0, u_1^{(1)}(0) = 0, u_1^{(2)}(0) = 0, u_1^{(3)}(0) = 0,$$

$$p^2 : u_2^{(4)}(x) = e^x u_1^{(3)} - u_1,$$

$$u_2(0) = 0, u_2^{(1)}(0) = 0, u_2^{(2)}(0) = 0, u_2^{(3)}(0) = 0,$$
$$\vdots$$

where, A, B, C and D are unknown constants to be determined. The corresponding solutions for the above system of equations are the series solution given as:

$$u_0(x) = \frac{1}{96}(-96 + 96e^{2x} - 3e^{3x} + e^x(3 + 96A + 6(-31 + 16B)x + 6(1 + 8C)x^2 + 4(-7 + 4D)x^3 + 2x^4))$$

$$u_1(x) = \frac{1}{1451520}(e^{-x}(1451520 + 93555e^{3x} - 4480e^{4x} + 362880e^{2x}(-19 + 4C + 2x) + e^x(5354125$$

$$+7446810x + 1828890x^2 + 923580x^3 - 1890(-31 + 32A)x^4 - 756(-31 + 16B)x^5 - 252(1$$

$$+8C)x^6 + 504x^7 - 18x^8 - 288D(5040 + 5040x + 2520x^2 + 840x^3 + x^7))))$$
$$\vdots$$

Using only 6-term approximation that is:

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_5(x) \tag{15}$$

Imposing the boundary conditions of the system (5) on Eq. (15) the values of the constants A, B, C and D can be obtained as:

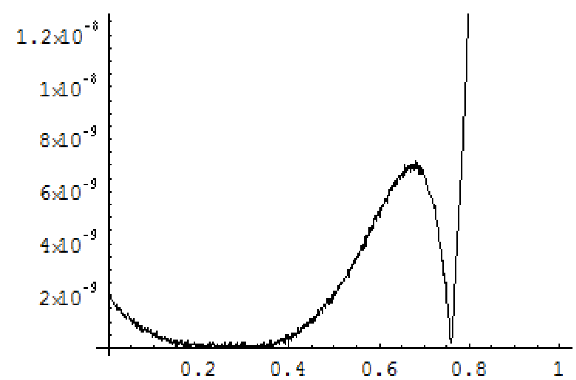

Fig. 2: Plot of errors

$$A = 0.9999999980259633,$$

$$B = 1.0000000216759806,$$

$$C = -1.6366491839105507 \times 10^{-7},$$

$$D = 1.00000056811826.$$

Then, the series solution can be expressed as:

$$U(x) = 1 + x - 8.16726 \times 10^{-8} x^2 + 0.16667x^3 + 2.38316 \times 10^{-8} x^4 + 0.00833334x^5 + 4.15407 \times 10^{-9} x^6$$

$$+0.000198414x^7 + 7.15518 \times 10^{-10} x^8 2.75604 \times 10^{-6} x^9 + 1.2948 \times 10^{-10} x^{10} - 02.49988$$

$$\times 10^{-8} x^{11} - 1.31503 \times 10^{-8} x^{12} + O(x^{13}). \tag{16}$$

The approximate series solution of the problem (2) is compared with $u(x) = 1 + \sinh(x)$ (Lin and Lin, 2010; Wu and Li, 2011) in Table 2, which shows that the method is quite efficient. Absolute errors $|U - u_{Exact}|$ are plotted in Fig. 2.

Example 3: The following fourth order nonlinear boundary value problem is considered:

$$\left.\begin{aligned} &u^{(4)}(x) - e^{-x}u^2(x) = 0, 0 \le x \le 1 \\ &u(0) = u^{(1)}(0) = 1, u\left(\frac{3}{4}\right) = e^{\frac{3}{4}}, u(1) = e. \end{aligned}\right\} \tag{17}$$

Table 3: Comparison of numerical results for problem (3)

x	Exact solution	Approximate series solution	Absolute error present method
0.0	1.00000	1.00000	6.26543E-12
0.1	1.10517	1.10517	2.55342E-09
0.2	1.22140	1.22140	8.60246E-09
0.3	1.34986	1.34986	1.57141E-08
0.4	1.49182	1.49182	2.15020E-08
0.5	1.64872	1.64872	2.35332E-08
0.6	1.82212	1.82212	1.96291E-08
0.7	2.01375	2.01375	8.27396E-09
0.8	2.22554	2.22554	9.18081E-09
0.9	2.45960	2.45960	2.28539E-08
1.0	2.71828	2.71828	8.86402E-12

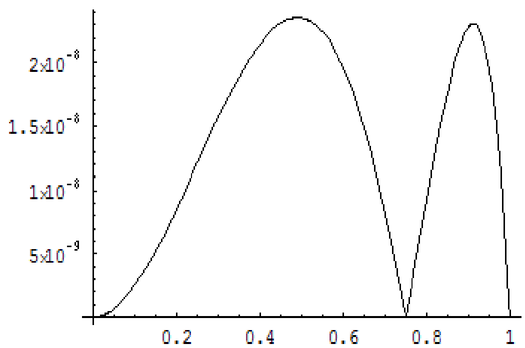

Fig. 3: Plot of absolute errors

The exact solution of the problem (3) is $u(x) = e^x$

Using the homotopy perturbation method, the following homotopy for the system (17) is constructed:

$$u^{(4)}(x) = p[e^{-x}u^2], \tag{18}$$

where, $p \in [0,1]$ is the embedding parameter. Assume that the solution of the given problem is:

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{19}$$

The nonlinear term $N(u)$ in Eq. (18) can be expressed as:

$$N(u) = N(u_0) + pN(u_0,u_1) + p^2N(u_0,u_1,u_2) + \cdots, \tag{20}$$

where,

$$N(u_0,u_1,\dots,u_n) = \frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{k=0}^{n} p^k u_k\right)\right]_{p=0},$$

$$n = 0,1,2,\dots$$

is called He's polynomial (Ghorbani, 2009) Substituting Eq. (19) and (20) in Eq. (18) and equating the coefficients of like powers of p, gives the following set of differential equations:

$$p^0 : u_0^{(4)}(x) = 0,$$

$$u_0(0) = 1, u_0^{(1)}(0) = 1, u_0^{(2)}(0) = A, u_0^{(3)}(0) = B,$$

$$p^1 : u_1^{(4)}(x) = e^{-x}u_0^2,$$
$$u_1(0) = 0, u_1^{(1)}(0) = 0, u_1^{(2)}(0) = 0, u_1^{(3)}(0) = 0,$$

$$p^2 : u_2^{(4)}(x) = e^{-x}u_0u_1,$$

$$u_2(0) = 0, u_2^{(1)}(0) = 0, u_2^{(2)}(0) = 0, u_2^{(3)}(0) = 0,$$
$$\vdots$$

where, A and B are unknown constants to be determined. Following Example (1), using the 3-term approximation and imposing the boundary conditions at $x = 0.75$ and $x = 1$, the constants are obtained as:

$$A = 0.9999994087690695,$$

$$B = 1.0000024198861392.$$

Then, the series solution can be expressed as:

$$U(x) = 1 + x - 0.5x^2 + 0.166667x^3 + 0.416667x^4 + 0.008333337x^5 + 0.00138889x^6$$
$$+ 0.000198414x^7 + 0.0000248016x^8 + 2.75573\times10^{-6}x^9 + 0.75571\times10^{-7}x^{10}$$
$$+ 2.50527\times10^{-8}x^{11} - 1.524\times10^{-7}x^{12} + O(x^{13}).$$

In Table 3, the comparison of the exact solution with the series solution of the problem (3) is given, which shows that the method is quite efficient. In Fig. 3 absolute errors $|U - u_{Exact}|$ are plotted in Fig. 3.

Example 4: The following fifth order nonlinear three point's boundary value problem is considered:

$$\left.\begin{aligned} &u^{(5)}(x) - e^{-x}u^2(x) = 0, 0 < x < 1 \\ &u(0) = u^{(1)}(0) = 1, u\left(\frac{1}{2}\right) = e^{\frac{1}{2}}, u(1) = u^{(2)}(1) = e. \end{aligned}\right\} \tag{21}$$

The exact solution of the problem (4) is $u(x) = e^x$.

Using the homotopy perturbation method, the following homotopy for the system (21) is constructed:

$$u^{(5)}(x) = p[e^{-x}u^2], \tag{22}$$

where, $p \in [0,1]$ is the embedding parameter. Assume that the solution of the given problem is:

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{23}$$

The nonlinear term $N(u)$ in Eq. (18) can be expressed as:

Table 4: Comparison of numerical results for problem (4)

x	Exact solution	Approximate series solution	Absolute error present method
0.0	1.00000	1.00000	0.000000
0.1	1.10517	1.10517	5.58569E-10
0.2	1.2214	1.22140	3.80139E-10
0.3	1.34986	1.34986	4.51430E-10
0.4	1.49182	1.49182	2.60672E-10
0.5	1.64872	1.64872	2.39371E-10
0.6	1.82212	1.82212	7.77565E-11
0.7	2.01375	2.01375	1.64396E-10
0.8	2.22554	2.22554	8.80967E-10
0.9	2.45960	2.45960	1.39270E-10
1.0	2.71828	2.71828	2.48480E-10

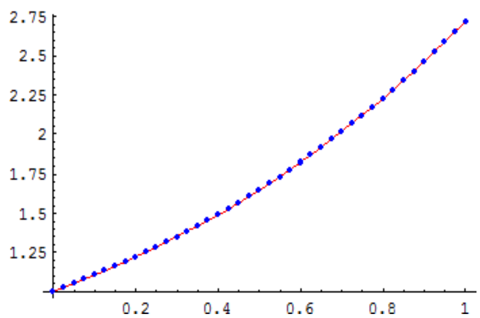

Fig. 4: Comparison of the approximate solution with the exact solution for problem (4). Dotted line: Approximate solution, solid line: the exact solution

$$N(u) = N(u_0) + pN(u_0, u_1) + p^2 N(u_0, u_1, u_2) + \cdots, \tag{24}$$

where,

$$N(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{d^n}{dp^n} \left[N\left(\sum_{k=0}^{n} p^k u_k \right) \right]_{p=0}, \quad n = 0, 1, 2, \ldots$$

is called He's polynomial (Ghorbani, 2009). Substituting Eq. (23) and (24) in Eq. (22) and equating the coefficients of like powers of p, gives the following set of differential equations:

$$p^0 : u_0^{(5)}(x) = 0,$$

$$u_0(0) = 1, u_0^{(1)}(0) = 1, u_0^{(2)}(0) = A, u_0^{(3)}(0) = B, u_0^{(4)}(0) = C,$$

$$p^1 : \; u_1^{(5)}(x) = e^{-x} u_0^2,$$

$$u_1(0) = 0, u_1^{(1)}(0) = 0, u_1^{(2)}(0) = 0, u_1^{(3)}(0) = 0, u_1^{(4)}(0) = 0,$$

$$p^2 : u_2^{(5)}(x) = e^{-x} u_0 u_1,$$
$$u_2(0) = 0, u_2^{(1)}(0) = 0, u_2^{(2)}(0) = 0, u_2^{(3)}(0) = 0, u_2^{(4)}(0) = 0,$$
$$\vdots$$

where, A, B and C are unknown constants to be determined. Following Example (1), using the 3-term approximation and imposing the boundary conditions at $x = 0.75$ and $x = 1$, the constants are obtained as:

$$A = 1.00000000568, \; B = 0.99999994805,$$

$$C = 1.00000014256.$$

Then, the series solution can be expressed as:

$$U(x) = 1 + x - 05000000028x^2 + 0.166666x^3 + 0.4166667x^4 + 0.008333333x^5 + 0.001388889x^6$$
$$+ 0.00019841x^7 + 0.00002480x^8 + 2.7557327 \times 10^{-6} x^9 + 2.7557319 \times 10^{-7} x^{10}$$
$$+ 2.50521 \times 10^{-8} x^{11} + 2.087675 \times 10^{-9} x^{12} + O(x^{13}).$$

In Table 4, the comparison of the exact solution with the series solution of the problem (4) is given, which shows that the method is quite efficient. In Fig. 4 absolute errors $|U - u_{Exact}|$ are plotted.

Example 5: The following sixth order nonlinear boundary value problem is considered:

$$\left. \begin{array}{l} u^{(6)}(x) - e^{-x} u^2(x) = 0, 0 < x < 1 \\ u(0) = u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = 1, u\left(\frac{1}{2}\right) = e^{\frac{1}{2}}, u(1) = e. \end{array} \right\} \tag{25}$$

The exact solution of the problem (5) is $u(x) = e^x$.

Using the aforesaid method, the series solution can be expressed as:

$$U(x) = 1 + (1.)x - 0.5x^2 + 0.166667x^3 + 0.4166667x^4 + 0.008333333x^5 + 0.00138885x^6$$
$$+ 0.000198432x^7 + 0.0000247952x^8 + 2.75728 \times 10^{-6} x^9 + 2.75381 \times 10^{-7} x^{10}$$
$$+ 2.49973 \times 10^{-8} x^{11} + 2.14303 \times 10^{-9} x^{12} + O(x^{13}).$$

The comparison of the exact solution with the series solution of the problem (5) is given in Table 5, which shows that the method is quite accurate.

Example 6: The following seventh order nonlinear boundary value problem is considered:

$$\left. \begin{array}{l} u^{(7)}(x) - e^{-x} u^2(x) = 0, 0 < x < 1 \\ u(0) = u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = u^{(4)}(0) = 1, u\left(\frac{1}{2}\right) = e^{\frac{1}{2}}, u(1) = e. \end{array} \right\} \tag{26}$$

The exact solution of the problem (6) is $u(x) = e^x$.

Using the aforesaid method, the series solution can be expressed as:

$$U(x) = 0.999998 + (1.)\,x - 0.499998x^2 + 0.166668x^3 + 0.4166661x^4 + 0.00833367x^5 + 0.00138876x^6$$
$$+ 0.000198417x^7 + 0.0000248361x^8 + 2.72677 \times 10^{-6} x^9 + 2.89152 \times 10^{-7} x^{10}$$
$$+ 2.14384 \times 10^{-8} x^{11} + 2.0249 \times 10^{-9} x^{12} + O(x^{13}).$$

The comparison of the exact solution with the series solution of the problem (6) is given in Table 6, which shows that the method is quite accurate.

Table 5: Comparison of numerical results for problem (5)

x	Exact solution	Approximate Series solution	Absolute error present method
0.0	1.00000	1.00000	7.77951E-09
0.1	1.10517	1.10517	1.16784E-08
0.2	1.22140	1.22140	7.57914E-09
0.3	1.34986	1.34986	2.04205E-08
0.4	1.49182	1.49182	1.75262E-08
0.5	1.64872	1.64872	1.03601E-08
0.6	1.82212	1.82212	1.60579E-09
0.7	2.01375	2.01375	4.20526E-10
0.8	2.22554	2.22554	2.25408E-08
0.9	2.45960	2.45960	8.26443E-09
1.0	2.71828	2.71828	1.69864E-08

Table 6: Comparison of numerical results for problem (6)

x	Exact solution	Approximate series solution	Absolute error present method
0.0	1.00000	1.00000	7.53520E-09
0.1	1.10517	1.10517	5.25690E-07
0.2	1.22140	1.22140	6.70140E-07
0.3	1.34986	1.34986	1.66395E-06
0.4	1.49182	1.49182	1.38077E-07
0.5	1.64872	1.64872	1.15557E-07
0.6	1.82212	1.82212	4.62997E-07
0.7	2.01375	2.01375	7.00576E-07
0.8	2.22554	2.22554	1.52829E-06
0.9	2.45960	2.45960	2.48422E-07
1.0	2.71828	2.71828	6.29186E-07

Table 7: Comparison of numerical results for Example (7)

x	Exact solution	Approximate series solution	Absolute Error Present method
0.0	0.0000	0.0000	0.0000
0.1	0.9946	0.9946	5.69961E-14
0.2	0.1954	0.1954	8.93730E-15
0.3	0.2835	0.2835	4.05231E-15
0.4	0.3580	0.3580	1.54876E-14
0.5	0.4122	0.4122	1.45550E-133
0.6	0.4373	0.4373	1.03195E-13
0.7	0.4229	0.4229	4.16889E-14
0.8	0.3561	0.3561	2.33036E-13
0.9	0.2214	0.2214	2.39697E-13
1 . 0	0.0000	-2.1729E-09	2.17290E-13

Example 7: The following seventh order nonlinear boundary value problem is considered:

$$\left.\begin{array}{l} u^{(7)}(x) = -u(x) - e^{x}(35 + 12x + 2x^{2}), 0 \le x \le 1 \\ u(0) = 0, u^{(1)}(0) = 1, u^{(2)}(0) = 0, u^{(3)}(0) = -3, u^{(4)}(0) = -8, \\ u\left(\frac{1}{2}\right) = \frac{e^{\frac{1}{2}}}{4}, u(1) = e. \end{array}\right\} \tag{27}$$

The exact solution of the problem (7) is $u(x) = x(1-x)e^{x}$. Using the aforesaid method, the series solution can be expressed as:

$$\begin{aligned} U(x) &= x - 0.5x^{3} - 0.333333x^{4} - 0.125x^{5} - 0.333333x^{6} - 0.00694444x^{7} - 0.00119048x^{8} \\ &- 0.000173611x^{9} - 0.0000220459x^{10} - 2.48016\times10^{-6}x^{11} - 2.50521\times10^{-7}x^{12} + O(x^{13}). \end{aligned}$$

The comparison of the exact solution with the series solution of the problem (7) is given in Table 7, which shows that the method is quite accurate.

CONCLUSION

In this study, the homotopy perturbation method has been applied to solve the multi-point boundary value problems. It is clearly seen that homotopy method is a powerful and accurate method for finding solutions for multi-point boundary value problems in the form of analytical expressions and presents a rapid convergence for the solutions. The numerical results showed that the homotopy perturbation method can solve the problem effectively and the comparison shows that the present method is in good agreement with the existing results in the literature.

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REFERENCES

Akram, G. and H.U. Rehman, 2013a. Numerical solution of eighth order boundary value problems in reproducing kernel space. Numer. Algorithms, 62(3): 527-540.

Akram, G. and H.U. Rehman, 2013b. Solutions of a Class of Sixth Order Boundary Value Problems Using the Reproducing Kernel Space. Abstr. Appl. Anal., Vol. 2013, Article ID 560590, pp: 8. Retrieved from: http://dx.doi.org/10.1155/2013/560590.

Akram, G., M. Tehseen, S.S. Siddiqi and H.U. Rehman, 2013c. Solution of a linear third order multi-point boundary value problem using RKM, Brit. J. Math. Comput. Sci. 3(2):180-194.

Ali, J., S. Islam, S. Islam and G. Zaman, 2010. The solution of multipoint boundary value problems by the optimal homotopy asymptotic method. Comput. Math. Appl., 59: 2000-2006.

Belndez, A., C. Pascual, A. Marquez and D.I. Mendez, 2007. Application of Hes homotopy perturbation method to the relativistic (an) harmonic oscillator. I: Comparison between approximate and exact frequencies. Int. J. Nonlin. Sci. Num., 8(4): 483-492.

Biazar, J. and H. Ghazvini, 2009. Convergence of the homotopy perturbation method for partial differential equations. Nonlinear Anal-Real, 10: 2633-2640.

Chun, C. and R. Sakthivel, 2010. Homotopy perturbation technique for solving two- point boundary value problems-comparison with other methods. Comput. Phy. Commun., 181: 1021-1024.

Dehghan, M. and F. Shakeri, 2008. Use of He’s homotopy perturbation method for solving a partial differential equation arising in modelling of flow in porous media. J. Porous Media, 11: 765-778.

Eloe, P.W. and J. Henderson, 2007. Uniqueness implies existence and uniqueness conditions for nonlocal boundary value problems for nth order differential equations. J. Math. Anal. Appl., 331: 240-247.

Feng, W. and. J.R.L. Webb, 1997. Solvability of m-point boundary value problems with non-linear growth. J. Math. Anal. Appl., 212: 467-480.

Geng, F., 2009. Solving singular second order three-point boundary value problems using re-producing kernel Hilbert space method. Appl. Math. Comput., 215: 2095-2102.

Geng, F. and M. Cui, 2010. Multi-point boundary value problem for optimal bridge design. Int. J. Comput. Math., 87: 1051-1056.

Ghorbani, A., 2009. Beyond adomian polynomials: He's polynomials, Chaos Soliton. Fract., 39: 1486-1492.

Graef, J.R. and J.R.L. Webb, 2009. Third order boundary value problems with nonlocal boundary conditions. Nonlinear Anal., 71: 1542-1551.

Hajji, M.A., 2009. Multi-point special boundary-value problems and applications to fluid flow through porous media. Proceeding of the International Multi-Conference of Engineers and Computer Scientists. Hong Kong. II.

He, J.H., 1999. Homotopy perturbation technique. Comput. Meth. Appl. Mech. Eng., 178: 257-262.

He, J.H., 2003. Homotopy perturbation method: A new nonlinear analytical technique. Appl. Math. Comput., 1: 73-79.

He, J.H., 2004. The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl. Math. Comput., 151(1): 278-292.

He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems. Int. J. Nonlin. Sci. Num., 6(2): 207-208.

He, J.H., 2006. Non-Perturbative Methods for Strongly Nonlinear Problems. Diss., de-Verlag in GmbH, Berlin.

Henderson, J. and C.J. Kunkel, 2008. Uniqueness of solution of linear nonlocal boundary value problems, Appl. Math. Lett., 21: 10531056.

Hussein, A.J., 2011. Study of error and convergence of homotopy perturbation method for two and three dimension linear Schrödinger equation. J. Coll. Educ., 1(2): 21-43.

Lin, Y.Z. and J.N. Lin, 2010. Numerical algorithm about a class of linear nonlocal boundary value problems. Appl. Math. Lett., 23: 997-1002.

Liu, B., 2003. Solvability of multi-point boundary value problem at resonance-Part IV. Appl. Math. Comput., 143: 275-199.

Moshiinsky, M., 1950. Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinues. Bol. Soc. Mat. Mexicana, 7: 1-25.

Rana, M.A., A.M. Siddiqui, Q.K. Ghori and R. Qamar, 2007. Application of He's homotopy perturbation method to sumudu transforms. Int. J. Nonlin. Sci. Num., 8(2): 185-190.

Saadatmandi, A. and M. Dehghan, 2012. The use of sinc-collocation method for solving multi-point boundary value problems. Commun. Nonlinear Sci., 17: 593-601.

Siddiqi, S.S. and G. Akram, 2006a. Solutions of fifth order boundary-value problems using nonpolynomial spline technique. Appl. Math. Comput., 175(2): 1574-1581.

Siddiqi, S.S. and G. Akram, 2006b. Solutions of sixth order boundary-value problems using nonpolynomial spline technique. Appl. Math. Comput., 181: 708-720.

Siddiqi, S.S. and M. Iftikhar, 2013a. Solution of seventh order boundary value problems by variation of parameters method. Res. J. Appl. Sci. Eng. Technol., 5(1):176-179.

Siddiqi, S.S. and M. Iftikhar, 2013b. Numerical solutions of higher order boundary value problems, Abstr. Appl. Anal., Vol. 2013, Article ID 427521, pp: 12. Retrieved from: http://dx.doi.org/10.1155/2013/427521.

Siddiqi S.S., G. Akram and M. Iftikhar, 2012a. Solution of seventh order boundary value problems by variational iteration technique, Appl. Math. Sci., 6(93-96): 4663-4672.

Siddiqi S.S., G. Akram and M. Iftikhar, 2012b. Solution of seventh order boundary value problem by differential transformation method. World Appl. Sci. J., 16(11): 1521-1526.

Tatari, M. and M. Dehghan, 2006. The use of the Adomian decomposition method for solving multipoint boundary value problems. Phys. Scripta., 73: 672-676.

Timoshenko, S., 1961. Theory of Elastic Stability. McGraw-Hill, New York.

Tirmizi, I.A., E.H. Twizell and S. Islam, 2005. A numerical method for third-order non-linear boundary-value problems in engineering. Int. J. Comput. Math., 82(1): 103-109.

Turkyilmazoglu, M., 2011. Convergence of the homotopy perturbation method. Int. J. Nonlin. Sci. Num., 12: 9-14.

Wu, B.Y. and X.Y. Li, 2011. A new algorithm for a class of linear nonlocal boundary value problems based on the reproducing kernel method. Appl. Math. Lett., 24: 156-159.

Yusufoglu, E., 2007. Homotopy perturbation method for solving a nonlinear system of second order boundary value problems. Int. J. Nonlin. Sci. Num., 8(3): 353-358.