## Research Article
# A Numerical Solution for One-dimensional Parabolic Equation Using Pseudo-spectral Integration Matrix and FDM

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**Abstract:** This study presents a numerical method for the solution of one type of PDEs equation. In this study, apply the pseudo-spectral successive integration method to approximate the solution of the one-dimensional parabolic equation. This method is based on El-Gendi pseudo-spectral method. Also the Finite Difference Method (FDM) is used as a minor method. The present numerical results are in satisfactory agreement with exact solution.

**Keywords:** El-Gendi method, Gauss-Lobatto points. pseudo-spectral successive integration, parabolic equation

## INTRODUCTION

Clenshaw and Curtis (1960) presented a new numerical method for integration of a function based on the Chebyshev polynomials. El-Gendi (1969) developed a new numerical scheme based on the Clenshaw and Curtis quadrature scheme which described a new method for the numerical solution of linear integral equations of Fredholm type and of Voltera type. This method has been extended to the linear integro-differential equations and ordinary differential equations. This method is presented an operation matrix for integration.

Delves and Mohamed (1985) used the El-Gendi method to solving the integral equations. They have shown that the El-Gendi (1975) method represented a modification of the Nystrom scheme when applied to solving second kind of Fredholm integral equations. Also, in Jeffreys and Jeffreys (1956) suggested a new method to solving the differential equations based on the successive integration of the Chebyshev expansions. This method is accomplished by starting with Chebyshev approximations for the highest order derivative and generating approximations to the lower order derivatives through successive integration of the highest order derivative. Hatziavramidis and Ku (1985) have been presented a Chebyshev expansion method for the solution of boundary-value problems of O.D.E type by using the pseudo-spectral successive integration method. The method is easier to implement than spectral methods employing the Galerkin and Tau approximations and yields results of comparable accuracy to these methods, with reduce computing requirement. Also Nasr et al. (1990) and Nasr and El-Hawary (1991) respectively, used the El-Gendi method and successive integration method to solving the

Falkner-skan equation which uses a boundary value technique and the Orr-sommerfeld equation for both plane poiseulle flow and the Blasius velocity profile.

The authors of El-Gendi et al. (1992) presentedan operation matrix for the successive integration. In fact, this matrix is generalization of the El-Gendi matrix. Elbarbary (2007) presented a modification of the El-Gendi successive integration matrix in (El-Gendi et al., 1992) which yields more accurate results than those computed by El-Gendi matrix in solving problems. Finally, Elgindy (2009) developed a new explicit expression of the higher order pseudo-spectral integration matrices. Applications to initial value problems, boundary value problems and linear integral and integro-differential equations are presented.

The purpose of this study is to apply the Pseudo-spectral successive integration method to solving the one-dimensional parabolic equation. The Pseudo-spectral successive integration method in based on El-Gendi (1969) Also; the finite difference method is used.

**El-Gendi's method:** Clenshaw and Curtis (1960) presented the numerical integration of a 'well-behaved' function $f(x)$ in $-1 \leq x \leq 1$, by Chebyshev expansion of $f(x)$ as follows:

$$f(x) = \sum_{r=0}^{''N} a_r T_r(x), \tag{1}$$

where:

$$a_r = \frac{2}{N} \sum_{j=0}^{''N} f(x_j) T_r(x_j) \tag{2}$$

and,

$$x_j = -\cos \frac{j\pi}{N}, j = 0,1,\dots,N \tag{3}$$

and integrating this series term by term. A summation symbol with double prims denotes a sum with first and last terms halved. The operation matrix B to approximate the indefiniteintegral $\int_{-1}^{x} f(t)\, dt$, presented by El-Gendi (1969) as follows:

$$\int_{-1}^{x} f(t)\, dt = \sum_{j=0}^{N} a_r \int_{-1}^{x} T_j(t)\, dt = \sum_{r=0}^{N+1} c_r\, T_r(x) \tag{4}$$

where,

$$\begin{cases} c_0 = \sum_{\substack{j=0 \\ j \neq 1}}^{''N} \frac{(-1)^{j+1} a_j}{j^2-1} - \frac{1}{4} a_1, \\[2mm] c_k = \frac{a_{k-1}-a_{k+1}}{2k}, \quad k = 1,2,\dots,N-2 \\[2mm] c_{N-1} = \frac{a_{N-2}-\frac{1}{2}a_N}{2(N-1)}, \\[2mm] c_N = \frac{a_{N-1}}{2N}, \\[2mm] \qquad c_{N+1} = \frac{a_N}{4(N+1)} \end{cases} \tag{5}$$

Then, after certain re-rearrangement we define the matrix B as follows:

$$\left[\int_{-1}^{x} f(t)\, dt\right] = B[f], \tag{6}$$

where, B is a square matrix of order N+1 and $[f] = [f(x_0), f(x_1), \dots, f(x_N)]^t$ where $x_j$ are Gauss-Lobatto points (3).

Finally, in El-Gendi *et al.* (1992) to approximate the successive integration of function $f(x)$ we have:

$$[I_n(f)] = \left[\int_{-1}^{x} \int_{-1}^{t_{n-1}} \int_{-1}^{t_{n-2}} \dots \int_{-1}^{t_2} \int_{-1}^{t_1} f(t_0)\, dt_0 dt_1 \dots dt_{n-2} dt_{n-1}\right] = B^{(n)}[f] \tag{7}$$

where,

$$B^{(n)} = \left[b_{i,j}^{(n)}\right], \tag{8}$$

$$b_{i,j}^{(n)} = \frac{(x_i - x_j)^{n-1}}{(n-1)!} b_{i,j}, i,j = 0(1)N \tag{9}$$

This study is organized as follows: In section 2, the pseudo-spectral integration method which is based on the El-Gendi (1975) method is presented. In section 3, the main problem is solved by using the present pseudo-spectral integration method and finite difference method. In section 4 for a given example with analytical solution, employed presented method and numerical results are presented. A brief conclusion is in section 5.

**Pseudo spectral integration matrix:** We assume that $(P_N f)(x)$ is Nth order Chebyshev interpolating polynomial of the function $f(x)$ in the points $(x_k, f(x_k))$ where,

$$(P_N f)(x) = \sum_{j=0}^{N} f_j\, \varphi_j(x) \tag{10}$$

with:

$$\varphi_j(x) = \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r\, T_r(x)\, T_r(x_j) \tag{11}$$

where, $\varphi_j(x) = \delta_{j,k}$ ($\delta_{j,k}$ is Kronecker delta) and $\alpha_0 = \alpha_N = \frac{1}{2}$, $\alpha_j = 1$ for $j = 1(1)N - 1$. Since $(P_N f)(x)$ is a unique Nth interpolating polynomial, it can be expressed in terms of a series expansion of the classical Chebyshev polynomials, hence we have:

$$(P_N f)(x) = \sum_{r=0}^{N} a_r\, T_r(x), \tag{12}$$

where,

$$a_r = \frac{2\alpha_r}{N} \sum_{j=0}^{N} \alpha_j\, f(x_j)\, T_r(x_j), \tag{13}$$

The successive integration of $f(x)$ at the points $x_k$ can be estimated by successive integration of $(P_N f)(x)$. Thus we have:

$$I_n(f) = \sum_{r=0}^{N} a_r \int_{-1}^{x} \int_{-1}^{t_{n-1}} \int_{-1}^{t_{n-2}} \dots \int_{-1}^{t_2} \int_{-1}^{t_1} T_r(t_0)\, dt_0 dt_1 \dots dt_{n-2} dt_{n-1}, \tag{14}$$

**Theorem 1:** Khalifa *et al.* (2003) the exact relation between Chebyshev functions and its derivatives is expressed as:

$$T_r(x) = \sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} T_{r+n-2m}^{(n)}(x), \quad r > n$$

where,

$$\chi_m = \prod_{\substack{j=0 \\ j \neq n-m}}^{n} (r + n - m - j)$$

Proof (Khalifa *et al.*, 2003).

**Theorem 2:** Elbarbary (2007) The successive integration of Chebyshev polynomials is expressed in terms of Chebyshev polynomials as follows:

$$\int_{-1}^{x} \int_{-1}^{t_{n-1}} \int_{-1}^{t_{n-2}} \dots \int_{-1}^{t_2} \int_{-1}^{t_1} T_r(t_0)\, dt_0 dt_1 \dots dt_{n-2} dt_{n-1} = \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x)$$

where,

$$\xi_{n,m,r}(x) = T_{r+n-2m}(x) - \sum_{i=0}^{n-1} \eta_i T_{r+n-2m}^{(i)}(-1),$$

$$\eta_i = \sum_{j=0}^{i} \frac{x^j}{(i-j)! j!}, \chi_m = \prod_{\substack{j=0 \\ j \neq n-m}}^{n} (r + n - m - j),$$

$$\beta_i = \begin{cases} 2 & i = 0 \\ 1 & i > 0 \end{cases}, \gamma_i = \begin{cases} n & i = 0 \\ n - i + 1 & 1 \leq i \leq n. \\ 0 & i > n \end{cases}$$

Proof (Elbarbary, 2007).
Thus, from Theorem 2 and (13), (14) we have:

$$I_n(f) = \sum_{j=0}^{N} \left( \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r T_r(x_j) \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x) \right) f(x_j)$$

The matrix form of the successive integration of the function $f(x)$ at the Gauss-Lobatto points $x_k$ is as follows:

$$[I_n(f)] = \left[ \sum_{j=0}^{N} \left( \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r T_r(x_j) \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x) \right) f(x_j) \right] = \Theta^{(n)}[f] \tag{15}$$

The elements of the matrix $\Theta^{(n)}$ are:

$$\theta_{k,j}^{(n)} = \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r T_r(x_j) \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x_k), \tag{16}$$

The matrix $\Theta^{(n)}$ which presented by ELbarbary (2007) is the pseudo-spectral integration matrix.

## ONE-DIMENSIONAL PARABOLIC EQUATION

We consider the one-dimensional parabolic equation (Dehghan and Tatari, 2006) of the form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)u(x,t) + F(x,t), 0 \leq x \leq 1, 0 \leq t \leq T \tag{17}$$

With initial condition:

$$u(x,0) = f(x), 0 \leq x \leq 1$$

And boundary conditions:

$$u(0,t) = g_0(t), u(1,t) = g_1(t), 0 \leq t \leq T,$$

With the over-specification at a point in the special domain:

$$u(x_0, t) = E(t), 0 \leq t \leq T$$

where, $g_0$, $g_1$, $F$ and $E$ are known functions and the functions $u$ and $p$ are unknowns. In this study, we apply both the pseudo spectral integration method and Finite Difference Method (FDM) (Strikverda, 2004) to solving this equation. (In this study, we suppose that the function $p$ is obtained by analytical method)

First, to translating $0 \leq x \leq 1$ to $-1 \leq X \leq 1$ we used $x = \frac{X+1}{2}$. Hence the Eq. (17) changed to the form:

$$\frac{\partial v}{\partial t} = 4\frac{\partial^2 v}{\partial X^2} + p(t)v(X,t) + Q(X,t), -1 \leq X \leq 1, 0 \leq t \leq T, \tag{18}$$

With initial condition:

$$v(X,0) = f(X), -1 \leq X \leq 1$$

And boundary conditions:

$$v(-1,t) = G_0(t), v(1,t) = G_1(t), 0 \leq t \leq T,$$

Also, with the over-specification at a point the special domain:

$$v(X_0, t) = K(t), 0 \leq t \leq T,$$

Now, we apply pseudo-spectral successive method and FDM to solving the Eq. (18) with its conditions.
We apply FDM on t-dimension, assume:

$$t_j = jh, \quad h = \frac{T}{n}, \quad j = 0(1)n$$

Hence, we have:

$$\frac{\partial v(X,t)}{\partial t}\bigg|_{t=t_0} = \frac{1}{2h}[-3v(X,t_0) + 4v(X,t_1) - v(X,t_2)] \tag{19}$$

$$\frac{\partial v(X,t)}{\partial t}\bigg|_{t=t_j} = \frac{1}{2h}[v(X,t_{j+1}) - v(X,t_{j-1})], j = 1(1)n-1 \tag{20}$$

$$\frac{\partial v(X,t)}{\partial t}\bigg|_{t=t_n} = \frac{1}{2h}[3v(X,t_n) - 4v(X,t_{n-1}) + v(X,t_{n-2})] \tag{21}$$

Substituting (19)-(21) into (18) gives:

$$-[3 + 2h\, p(t_0)]v(X,t_0) + 4v(X,t_1) - v(X,t_2) = 8h\frac{\partial^2 v(X,t_0)}{\partial X^2} + 2h\, Q(X,t_0), \tag{22}$$

$$v(X,t_{j+1}) - 2h\, p(t_j)v(X,t_j) - v(X,t_{j-1}) = 8h\frac{\partial^2 v(X,t_j)}{\partial X^2} + 2h\, Q(X,t_j), \tag{23}$$

$$[3 - 2h\, p(t_n)]v(X,t_n) - 4v(X,t_{n-1}) + v(X,t_{n-2}) = 8h\frac{\partial^2 v(X,t_n)}{\partial X^2} + 2h\, Q(X,t_n) \tag{24}$$

From the pseudo-spectral successive integration at the Gauss-Lobatto points $(x_i = -\cos\frac{i\pi}{N})$ we have:

$$\frac{\partial^2 v(X,t_j)}{\partial X^2}\bigg|_{X=X_i} = \varphi(X_i, t_j), \tag{25}$$

$$\frac{\partial v(X,t_j)}{\partial X}\bigg|_{X=X_i} = \sum_{k=0}^{N} \vartheta_{i,k}^{(1)} \varphi(X_k, t_j) + c_1, \tag{26}$$

$$v(X_i, t_j) = \sum_{k=0}^{N} \vartheta_{i,k}^{(2)} \varphi(X_k, t_j) + c_1(X_i + 1) + c_2 \tag{27}$$

$(for\ i = 0(1)N, \ j = 0(1)n)$.

The constants $c_1$ and $c_2$ are obtained to satisfy the boundary conditions. Thus:

$c_1 = -\frac{1}{2}\left(\sum_{k=0}^{N} \vartheta_{N,k}^{(2)} \varphi(X_k, t_j) - G_1(t_j) + G_0(t_j)\right)$, for $j = 0(1)n$

$c_2 = G_0(t_j)$, for $j = 0(1)n$

Substituting $c_1$ and $c_2$ into (27) gives us the approximation solution at point $(X_i, t_j)$ as follows:

$v(X_i, t_j) =$
$\sum_{k=0}^{N} \vartheta_{i,k}^{(2)} \varphi(X_k, t_j) - \frac{1}{2}(X_i + 1)\left(\sum_{k=0}^{N} \vartheta_{N,k}^{(2)} \varphi(X_k, t_j)\right) + \frac{1}{2}(X_i + 1)G_1(t_j) - \frac{1}{2}(X_i - 1)G_0(t_j)$ 　(28)

But, to complete our work, we need to find all the unknowns $\varphi_{i,j}$. Hence for this purpose we substitute (28) into (22)-(24) and solve the system of linear equations to present more details we define:

$A_i = [\vartheta_{i,0}^{(2)}, \vartheta_{i,1}^{(2)}, ..., \vartheta_{i,n}^{(2)}] - \frac{1}{2}(X_i + 1)[\vartheta_{N,0}^{(2)}, \vartheta_{N,1}^{(2)}, ..., \vartheta_{N,n}^{(2)}]$ 　(29)

$\Phi_j = [\varphi_{0,j}, \varphi_{1,j}, ..., \varphi_{N,j}]^t$ 　(30)

$Y_{i,j} = \frac{1}{2}(X_i + 1)G_1(t_j) - \frac{1}{2}(X_i - 1)G_0(t_j)$ 　(31)

Finally, if (29)-(31) substitute into (22)-(24), respectively then we have 3 systems of linear equations as:

- For $i = 0(1)N$ and $j = 0$

$-(3 + 2h\, p(t_0)) A_i \Phi_0 + 4 A_i \Phi_1 - A_i \Phi_2 - 8h\, \varphi_{i,0} - (3 + 2h\, p(t_0)) Y_{i,0} + 4 Y_{i,1} - Y_{i,2} - 2h\, Q(X, t_0) = 0$ 　(32)

- For $i = 0(1)N$ and $j = 1(1)n - 1$

$A_i \Phi_{j+1} - (2h\, p(t_j)) A_i \Phi_j - A_i \Phi_{j-1} - 8h\, \varphi_{i,j} + Y_{i,j+1} - (2h\, p(t_j)) Y_{i,j} - Y_{i,j-1} - 2h\, Q(X_i, t_j) = 0$ 　(33)

- For $i = 0(1)N$ and $j = n$

$(3 - 2h\, p(t_n)) A_i \Phi_n - 4 A_i \Phi_{n-1} + A_i \Phi_{n-2} - 8h\, \varphi_{i,n} + (3 - 2h\, p(t_n)) Y_{i,n} - 4 Y_{i,n-1} + Y_{i,n-2} - 2h\, Q(X, t_n) = 0$ 　(34)

Solving the systems (32)-(34) leads to obtaining the all unknowns $\varphi_{i,j}$. Notice that if $i = 0, N$ then $A_i = 0$, thus we can obtained the first and last rows of the matrix $\Phi$ as follows:

$\varphi_{0,0} = \frac{1}{8h}\left((3 + 2h\, p(t_0)) Y_{0,0} - 4 Y_{0,1} + Y_{0,2} + 2h\, Q(X_0, t_0)\right)$

$\varphi_{0,j} = \frac{1}{8h}\left(-Y_{0,j+1} + 2hp(t_j)Y_{0,j} + Y_{0,j-1} + 2hQ(X_0, t_j)\right)$
$j = 1(1)n - 1$

$\varphi_{0,n} = \frac{1}{8h}\left(-(3 - 2h\, p(t_n)) Y_{0,n} + 4 Y_{0,n-1} - Y_{0,n-2} + 2h\, Q(X_0, t_n)\right)$

and:

$\varphi_{N,0} = \frac{1}{8h}\left((3 + 2h\, p(t_0)) Y_{N,0} - 4 Y_{N,1} + Y_{N,2} + 2h\, Q(X_N, t_0)\right)$

$\varphi_{N,j} = \frac{1}{8h}\left(-Y_{N,j+1} + 2hp(t_j)Y_{N,j} + Y_{N,j-1} + 2hQ(X_N, t_j)\right), j = 1(1)n - 1$

$\varphi_{N,n} = \frac{1}{8h}\left(-(3 - 2hp(t_n))Y_{N,n} + 4 Y_{N,n-1} - Y_{N,n-2} + 2hQ(X_N, t_n)\right),$

Also, to computing another $\varphi_{i,j}$ for $i = 1(1)N - 1$ and $j = 0(1)n$ we always have the system of form:

$A\Phi^* - 8h\varphi_{i,j} = B$ 　(35)

where, A is 3-D matrix:

$A = $
$$\begin{bmatrix} -(3 + 2hp(t_0))A_i & 4A_i & -A_i & 0 & \cdots & & 0 \\ -A_i & -(2hp(t_1))A_i & A_i & 0 & \cdots & & 0 \\ & \vdots & & & \ddots & & \vdots \\ & 0 & & & \cdots & 0 & -A_i & -(2hp(t_{n-1}))A_i & A_i \\ & 0 & & & \cdots & & A_i & -4A_i & (3 - 2hp(t_n))A_i \end{bmatrix}$$
　(36)

and:

$\Phi^* = [\Phi_0, \Phi_1, \Phi_2, ..., \Phi_n]^t$ 　(37)

$$B = \begin{bmatrix} (3 + 2hp(t_0))Y_{i,0} - 4Y_{i,1} + Y_{i,2} + 2hQ(X_i, t_0) \\ -Y_{i,2} + (2hp(t_1))Y_{i,1} + Y_{i,0} + 2hQ(X_i, t_1) \\ -Y_{i,3} + (2hp(t_2))Y_{i,2} + Y_{i,1} + 2hQ(X_i, t_2) \\ \vdots \\ -Y_{i,n} + (2hp(t_{n-1}))Y_{i,n-1} + Y_{i,n-2} + 2hQ(X_i, t_n) \\ -(3 - 2hp(t_n))Y_{i,n} + 4Y_{i,n-1} + Y_{i,n-2} + 2hQ(X_i, t_n) \end{bmatrix}$$
　(38)

Finally, by solving (35), all unknowns $\varphi_{i,j}$ to be determined. Hence we can approximate the solutions of the main problem at all points $(X_i, t_j)$ by substituting $\varphi_{i,j}$ into (28).

Table 1: Absolute error for n = 4, N = 4 at points $(X_i, t_j)$

| X/ t | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $t_0$ | $5.2 \times 10^{-3}$ | $1.1 \times 10^{-2}$ | $4.8 \times 10^{-3}$ |
| $t_1$ | $8.5 \times 10^{-4}$ | $8.3 \times 10^{-4}$ | $1.6 \times 10^{-3}$ |
| $t_2$ | $3.1 \times 10^{-3}$ | $3.8 \times 10^{-3}$ | $2.6 \times 10^{-4}$ |
| $t_3$ | $9.4 \times 10^{-4}$ | $8.0 \times 10^{-4}$ | $4.8 \times 10^{-4}$ |
| $t_4$ | $2.1 \times 10^{-2}$ | $1.3 \times 10^{-2}$ | $1.1 \times 10^{-3}$ |

The absolute error at all points $(X_0, t_j)$ and $(X_4, t_j)$ for any $j = 0(1)4$ are equals to zero

Table 2: Absolute error for n = 8, N = 8 at points $(X_i, t_j)$

| X/ t | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| $t_0$ | $3.9\times10^{-4}$ | $1.4\times10^{-3}$ | $2.5\times10^{-3}$ | $2.8\times10^{-3}$ | $2.1\times10^{-3}$ | $1.1\times10^{-3}$ | $2.8\times10^{-4}$ |
| $t_1$ | $8.7\times10^{-5}$ | $3.4\times10^{-4}$ | $6.9\times10^{-4}$ | $8.8\times10^{-4}$ | $7.7\times10^{-4}$ | $4.4\times10^{-4}$ | $1.2\times10^{-4}$ |
| $t_2$ | $5.7\times10^{-5}$ | $1.7\times10^{-4}$ | $2.0\times10^{-4}$ | $9.4\times10^{-5}$ | $4.2\times10^{-5}$ | $8.3\times10^{-5}$ | $3.5\times10^{-5}$ |
| $t_3$ | $1.4\times10^{-4}$ | $4.7\times10^{-4}$ | $7.1\times10^{-4}$ | $6.6\times10^{-4}$ | $3.7\times10^{-4}$ | $1.2\times10^{-4}$ | $1.5\times10^{-5}$ |
| $t_4$ | $1.8\times10^{-4}$ | $6.2\times10^{-4}$ | $9.7\times10^{-4}$ | $9.5\times10^{-4}$ | $5.9\times10^{-4}$ | $2.2\times10^{-4}$ | $4.2\times10^{-5}$ |
| $t_5$ | $2.1\times10^{-4}$ | $7.0\times10^{-4}$ | $1.1\times10^{-3}$ | $1.1\times10^{-3}$ | $7.4\times10^{-4}$ | $3.1\times10^{-4}$ | $6.4\times10^{-5}$ |
| $t_6$ | $1.8\times10^{-4}$ | $6.1\times10^{-4}$ | $9.8\times10^{-4}$ | $9.8\times10^{-4}$ | $6.4\times10^{-4}$ | $2.7\times10^{-4}$ | $5.6\times10^{-5}$ |
| $t_7$ | $1.9\times10^{-4}$ | $6.7\times10^{-4}$ | $1.1\times10^{-3}$ | $1.2\times10^{-3}$ | $8.1\times10^{-4}$ | $3.6\times10^{-4}$ | $8.5\times10^{-5}$ |
| $t_8$ | $3.5\times10^{-5}$ | $6.3\times10^{-5}$ | $1.8\times10^{-5}$ | $1.7\times10^{-4}$ | $2.7\times10^{-4}$ | $2.2\times10^{-4}$ | $7.7\times10^{-5}$ |

The absolute error at all points $(X_0, t_j)$ and $(X_8, t_j)$ for any $j = 0(1)8$ are equals to zero.

Table 3: Absolute error for n = 10, N = 10 at points $(X_i, t_j)$

| X/t | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $t_0$ | $1.6\times10^{-4}$ | $6.2\times10^{-4}$ | $1.3\times10^{-3}$ | $1.7\times10^{-3}$ | $1.9\times10^{-3}$ | $1.6\times10^{-3}$ | $1.0\times10^{-3}$ | $4.8\times10^{-4}$ | $1.2\times10^{-4}$ |
| $t_1$ | $5.3\times10^{-5}$ | $2.1\times10^{-4}$ | $4.4\times10^{-4}$ | $6.6\times10^{-4}$ | $7.6\times10^{-4}$ | $6.8\times10^{-4}$ | $4.8\times10^{-4}$ | $2.4\times10^{-5}$ | $6.4\times10^{-5}$ |
| $t_2$ | $5.1\times10^{-6}$ | $8.4\times10^{-6}$ | $1.4\times10^{-5}$ | $7.2\times10^{-5}$ | $1.4\times10^{-4}$ | $1.8\times10^{-4}$ | $1.5\times10^{-4}$ | $9.1\times10^{-5}$ | $2.7\times10^{-5}$ |
| $t_3$ | $4.1\times10^{-5}$ | $1.4\times10^{-4}$ | $2.5\times10^{-4}$ | $2.9\times10^{-4}$ | $2.4\times10^{-4}$ | $1.3\times10^{-4}$ | $3.8\times10^{-5}$ | $3.7\times10^{-6}$ | $5.1\times10^{-6}$ |
| $t_4$ | $6.4\times10^{-5}$ | $2.3\times10^{-4}$ | $4.1\times10^{-4}$ | $5.1\times10^{-4}$ | $4.7\times10^{-4}$ | $3.2\times10^{-4}$ | $1.6\times10^{-4}$ | $5.0\times10^{-5}$ | $8.1\times10^{-6}$ |
| $t_5$ | $7.9\times10^{-5}$ | $2.8\times10^{-4}$ | $5.2\times10^{-4}$ | $6.6\times10^{-4}$ | $6.3\times10^{-4}$ | $4.5\times10^{-4}$ | $2.4\times10^{-4}$ | $8.8\times10^{-5}$ | $1.8\times10^{-5}$ |
| $t_6$ | $8.3\times10^{-5}$ | $3.0\times10^{-4}$ | $5.5\times10^{-4}$ | $7.1\times10^{-4}$ | $6.8\times10^{-4}$ | $5.0\times10^{-4}$ | $2.7\times10^{-4}$ | $1.0\times10^{-4}$ | $2.2\times10^{-5}$ |
| $t_7$ | $8.5\times10^{-5}$ | $3.1\times10^{-4}$ | $5.7\times10^{-4}$ | $7.5\times10^{-4}$ | $7.3\times10^{-4}$ | $5.5\times10^{-4}$ | $3.1\times10^{-4}$ | $1.3\times10^{-4}$ | $2.8\times10^{-5}$ |
| $t_8$ | $7.0\times10^{-5}$ | $2.5\times10^{-4}$ | $4.7\times10^{-4}$ | $6.2\times10^{-4}$ | $6.0\times10^{-4}$ | $4.5\times10^{-4}$ | $2.5\times10^{-4}$ | $1.0\times10^{-4}$ | $2.2\times10^{-5}$ |
| $t_9$ | $7.6\times10^{-5}$ | $2.8\times10^{-4}$ | $5.3\times10^{-4}$ | $7.1\times10^{-4}$ | $7.1\times10^{-4}$ | $5.4\times10^{-4}$ | $3.2\times10^{-4}$ | $1.4\times10^{-4}$ | $3.3\times10^{-5}$ |
| $t_{10}$ | $5.7\times10^{-6}$ | $3.2\times10^{-6}$ | $3.8\times10^{-5}$ | $1.2\times10^{-4}$ | $1.9\times10^{-4}$ | $2.3\times10^{-4}$ | $2.0\times10^{-4}$ | $1.2\times10^{-4}$ | $3.5\times10^{-5}$ |

The absolute error at all points $(X_0, t_j)$ and $(X_{10}, t_j)$ for any j = 0(1)10 are equals to zero
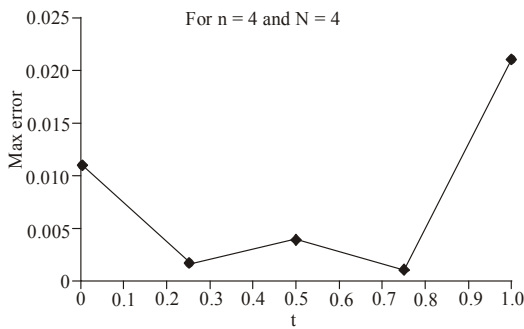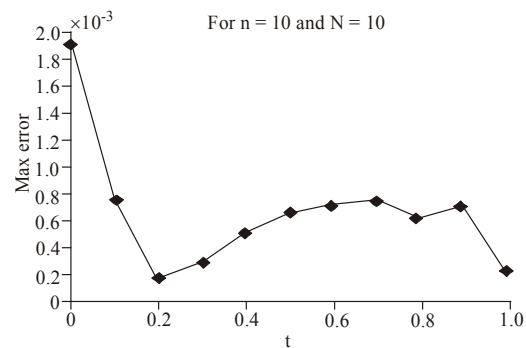


Fig. 1: The maximum error for n = 4 and N = 4
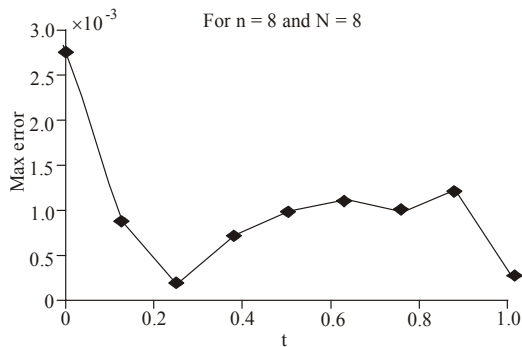


Fig. 2: The maximum error for n = 8 and N = 8



Fig. 3: The maximum error for n = 10 and N = 10

$$F(x,t) = (\pi^2 - (t + 1)^2)exp(-t^2)(\cos(\pi x) + \sin(\pi x))$$

$$E(t) = \sqrt{2}\exp(-t^2), x_0 = 0.25$$

where, $p(t) = 1 + t^2$ and the exact solution is:

$$u(x,t) = \exp(-t^2)(\cos(\pi x) + \sin(\pi x))$$

By presented approach in section 3 we solved this problem. The present numerical results are in the above Table 1 to 3 and Fig. 1 to 3.

**NUMERICAL RESULTS**

Consider the Eq. (17) in (14) with below details:

$$f(x) = \cos(\pi x) + \sin(\pi x)$$
$$g_0(t) = \exp(-t^2), g_1(t) = -\exp(-t^2)$$

**CONCLUSION**

This study applies the pseudo-spectral successive integration method to approximate the solutions of the one-dimensional parabolic equation at the points $(X_i, t_j)$. So far this method, don't applied to solve the

partial differential equations. This study, demonstrate that the pseudo-spectral successive integration method can solve the partial differential equations.

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