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Research Article

Numerical Solution of Seventh Order Boundary Value Problems Using the Reproducing Kernel Space

Ghazala Akram and Hamood Ur Rehman
Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan

Abstract: The aim of study of this article is to determine the solution of seventh order boundary value problem. The behavior of the induction motor is simulated by fifth order differential equation model and induction machine with two rotor circuits is represented by the seventh order differential equations. In this study, a Reproducing Kernel Method (RKM) for a class of seventh-order nonlinear boundary value problems is investigated. The argument is based on the reproducing kernel space W^8_2 [0, 1]. The proposed method gives better results when compared with the method available in literature (Siddiqi *et al*., 2012a). Two numerical examples are given to illustrate the implementation and efficiency of the method.

Keywords: Gram-schmidt orthogonal process, reproducing kernel, seventh order Boundary Value Problem (BVP)

INTRODUCTION

Commonly, 5th order differential equation model simulated the behavior of the induction motor which includes two stator state variables, two rotor state variables, and shaft speed. Two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage, or rotor distributed parameters. Generally, the induction machine with two rotor circuits is represented by the seventh order differential equations of flux linkages and speed (Richards and Sarma, 1994). Siddiqi and Akram (2006, 2007) presented non-polynomial spline method and sextic spline method for the numerical solution of the fifth-order linear special case boundary value problems. Siddiqi *et al*. (2007) developed quintic spline method for the numerical solutions of linear special case sixth-order boundary value problems. In Siddiqi *et al*. (2012a, b) and Siddiqi and Iftikhar (2013a) solutions of seventh order boundary value problems are discussed. Siddiqi and Iftikhar (2013b) presented the solution of higher order boundary value problems using the homotopy analysis method.

A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of differential equations (Akram and Rehman, 2011, 2013a, b, c, d; Geng and Cui, 2007; Li and Wu, 2013). In this study, a reproducing kernel method is used for the solution of linear and nonlinear seventh order BVP.

Consider the following seventh order two-point boundary value problem:

$$\left.\begin{array}{l} u^{(7)}(x) + a_0(x)\,u^{(4)}(x) + a_1(x)\,u^{(2)}(x) + a_2(x)\,u(x) = f(x, u(x)), \\ u^{(i)}(0) = \alpha_i,\ u^{(i)}(1) = \beta_i, i = 0,1,2,\ u^{(3)}(0) = \alpha_3 \quad 0 < x \le 1 \end{array}\right\} \quad (1)$$

where, $a_i(x), i = 0, 1, 2$ and $f(x, u(x))$ are continuous functions on [0, 1]. Let L be the differential operator and homogenization of the boundary conditions of system (1) can be transformed into the following form:

$$\left.\begin{array}{l} Lu(x) = f(x, u(x)), \\ u^{(i)}(0) = u^{(i)}(1) = 0, i = 0,1,2,\ u^{(3)}(0) = 0 \quad 0 < x \le 1 \end{array}\right\} \quad (2)$$

Thus, the solution of system (2) provides the solution of the system (1).

REPRODUCING KERNEL SPACES

The reproducing kernel space W^8_2 [0, 1] is defined by $W_2^8[01] = \{u(x)/u^{(i)}(x), i = 0,1,2...,7$ are absolutely continuous real valued functions in [0, 1], $u^{(8)}(x) \in L^2[0,1]\}$. The inner product and norm in W^8_2 [0, 1] are given by:

$$< u(x), v(x) >= \int_0^1 (u^{(7)}(x)v^{(7)}(x) + u^{(8)}(x)v^{(8)}(x))dx \quad (3)$$

$$\|u(x)\| = \sqrt{< u(x), v(x) >},\ u(x), v(x) \in W_2^8[0,1] \quad (4)$$

Theorem 1: The space W^8_2 [0, 1] is a reproducing kernel Hilbert space. That is, $\forall\, u(y) \in W_2^8[0,1]$ and each fixed $x, y \in [0,1]$, there exists $R_x(y) \in W_2^8[0,1]$ such that $< u(y), R_x(y) >= u(x)$ and

Corresponding Author: Ghazala Akram, Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan
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$R_x(y)$ is called the reproducing kernel function of space $W^8_2[0, 1]$.
The reproducing kernel function $R_x(y)$ is given by:

$$R_x(y) = \begin{cases} \sum_{i=0}^{13} c_i y^i + c_{14}e^y + c_{15}e^{-y}, & y \le x \\ \sum_{i=0}^{13} d_i y^i + d_{14}e^y + d_{15}e^{-y}, & y > x \end{cases} \quad (5)$$

THE EXACT AND APPROXIMATE SOLUTIONS

In the problem (2), the linear operator L: $W^8_2[0, 1] \to W^1_2[0, 1]$ is bounded. Using the adjoint operator L^* of L and choose a countable dense subset $T = \{x_1, x_2, \ldots, x_n, \ldots\} \subset [0,1]$ and let:

$$\varphi_i(y) = Q_{x_i}(y), i \in N \quad (6)$$

then $\psi_i(x) = L^*\varphi_i(x)$, where $\psi_i(x) \in W_2^8[0,1]$.

Lemma 1: $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W^8_2[0, 1]$ and $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof: For each fixed u (x) ∈ $W^8_2[0, 1]$, let $< u(x), \psi_i(x) >= 0, i = 1,2,\ldots$ which implies:

$$\langle u(x), (L^*\varphi_i)(x) \rangle =< (Lu)(x), Q_{x_i}(x) >= (Lu)(x_i) = 0$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0, 1]$, $(Lu)(x) = 0$, which implies $u = 0$ from the existence of L^{-1}.
Using reproducing property, it can be written as:

$$\psi_i(x) =< \psi_i(y), R_x(y) >=< (L^*\varphi_i)(y),$$
$$R_x(y) >=< (\varphi_i(y), LR_x(y)(x) >= L_y R_x(y)|_{y=x_i}$$

To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space $W^8_2[0, 1]$ Gram-Schmidt process can be used, as:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad i = 1,2,3,\ldots \quad (7)$$

Theorem 2: For all $u(x) \in W_2^8[0,1]$, the series $\sum_{i=0}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^8}$. On the other hand, if u (x) is the exact solution of the system (5) then:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k))\overline{\psi}_i(x)$$

Proof: Since $u(x) \in W_2^8[0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system as:

$$u(x) = \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x) \quad (8)$$

Since the space $W^8_2[0, 1]$ is Hilbert space so the series $\sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^8}$. From Eq. (7) and (8), it can be written as:

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} < u(x), \sum_{k=1}^{i} \beta_{ik}\psi_k(x) > \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < Lu(x), \varphi_k(x) > \overline{\psi}_i(x) \end{aligned}$$

If $u(x)$ is the exact solution of Eq. (2) and $Lu = f(x, u(x))$, then:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k))\overline{\psi}_i(x)$$

The approximate solution obtained by the n-term intercept of the exact solution $u(x)$, given by:

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k))\overline{\psi}_i(x) \quad (9)$$

The problem (2) is nonlinear, then approximate solution of the problem (2) can be obtained using the following iteration equation:

$$\begin{cases} \text{Any fixed } u_0(x) \in W_2^8[0,1] \\ u_n(x) = \sum_{i=1}^{n} A_i\overline{\psi}_i(x) \end{cases} \quad (10)$$

where,

$$\left.\begin{aligned} A_1 &= \beta_{11} f(x_1, u_0(x_1)) \\ A_2 &= \sum_{k=1}^{2} \beta_{2k} f(x_k, u_{k-1}(x_k)) \\ &\vdots \\ A_n &= \sum_{k=1}^{n} \beta_{nk} f(x_k, u_{k-1}(x_k)) \end{aligned}\right\} \quad (11)$$

Theorem 3: Let the following conditions are satisfied:

i. $\|u(x)\|$ is bounded and
ii. $\{x_i\}_{i=0}^{\infty}$ is dense in [0, 1]
iii. $f(x, u(x)) \in W_2^1[0,1]$ *and* $u(x) \in W_2^8[0,1]$ then $u_n(x)$ in Eq. (10) converges to the exact solution $u(x)$ of the problem (2), where A_i are given by Eq. (11) and:

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$$

Proof (i): First, we will prove the convergence of $u_n(x)$. From Eq. (12), it can be written as:

$$u_{n+1}(x) = u_n(x) + A_{n+1}\overline{\psi}_{n+1}(x)$$

Then the orthonormality of $\{\overline{\psi}(x)\}_{i=0}^{\infty}$ yields:

$$\|u_{n+1}(x)\|^2 = \|u_n(x)\|^2 + \|A_{n+1}\|^2 = \sum_{i=1}^{n+1} A_i$$

From boundedness of $\|u_n(x)\|$ gives $\sum_{i=1}^{\infty} A_i < \infty$ i.e., $\{A_i\} \in L^2, i = 1,2,...$
For,

$$m > n, (u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp (u_{n+1} - u_n)$$

leads to:

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + ... + u_{n+1} - u_n\|^2$$
$$= \left\|u_m - u_{m-1}\right\|^2 + \|u_{m-1} - u_{m-2}\|^2 + ... + \left\| u_{n+1} - u_n\right\|^2$$
$$= \sum_{i=n+1}^{m} (A_i)^2 \to 0, \ (n, m \to 0)$$

Considering the completeness of W^8_2 [0, 1], there exists $u(x) \in W_2^8[0,1]$, such that:

$$u_n(x) \to u(x), n \to \infty$$

(ii) Using (i) of Theorem 3, $u_n(x)$ converge uniformly to $u(x)$. On taking limits in Eq. (10), it follows that:

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$$

It may be noted that:

$$Lu(x_j) = \sum_{i=1}^{\infty} A_i < L\overline{\psi}_i(x), \varphi_j(x) >$$
$$= \sum_{i=1}^{\infty} A_i < \overline{\psi}_i(x), L^* \varphi_j(x) >$$
$$= \sum_{i=1}^{\infty} A_i < \overline{\psi}_i(x), \psi_j(x) >$$

Moreover,

$$\sum_{i=1}^{n} \beta_{nj} Lu(x_j) = \sum_{i=1}^{\infty} A_i < \overline{\psi}_i(x), \sum_{i=1}^{n} \beta_{nj} \psi_j(x) >$$
$$= \sum_{i=1}^{\infty} A_i < \overline{\psi}_i(x), \overline{\psi}_n(x) >= A_n$$

If $n = 1$, then:

$$Lu(x_1) = f(x_1, u_0(x_1))$$

If $n = 2$, then:

$$\beta_{21} Lu(x_1) + \beta_{22} Lu(x_2) = \beta_{21} f$$
$$(x_1, u_o(x_1)) + \beta_{22} f(x_2, u_1(x_2))$$

It is clear that:

$$Lu(x_2) = f(x_2, u_1(x_2))$$

Furthermore, it is easy to see by induction that:

$$Lu(x_j) = f(x_j, u_{j-1}(x_j)) \tag{12}$$

Since, $\{x\}_{i=1}^{\infty}$ is dense on interval [0, 1], for any $y \in$ [0, 1], there exists subsequence $\{x_{nj}\}$ such that:

$$x_{nj} \to y, \ y \to \infty$$

Let $y \to \infty$ in Eq. (12) and by the convergence of $u_n(x)$, gives:

$$Lu(x) = f(x, u(x)) \tag{13}$$

That is, $u(x)$ is the solution of the problem (2) and:

$$u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x) \tag{14}$$

where, A_i are given by Eq. (11). To illustrate the applicability and effectiveness of our method, two numerical examples are constructed.

Table 1: The comparisons of the errors in absolute values between the methods developed in this study and that of Siddiqi *et al*. (2012a)

x	Siddiqi *et al*. (2012a)	Present method n = 30	Present method n = 50
0.0	0.0000	0.0000	0.0000
0.1	3.0198 E-14	3.3085 E-14	2.7755 E-15
0.2	3.6903 E-13	2.0450 E-13	1.6875 E-14
0.3	1.3749 E-12	4.7584 E-13	3.9079 E-14
0.4	3.0308 E-12	7.2009 E-13	5.8397 E-14
0.5	4.7868 E-12	8.2056 E-13	6.6169 E-14
0.6	5.7388 E-12	7.2953 E-13	5.8287 E-14
0.7	5.1207 E-12	4.9066 E-13	5.8286 E-14
0.8	2.9893 E-12	2.1832 E-13	1.7930 E-14
0.9	6.9944 E-13	3.5860 E-14	1.1102 E-16
1.0	1.1102 E-16	4.4408 E-16	4.4409 E-16

Table 2: Error in absolute values obtained by the present method

x	Present method n = 30	Present method n = 50
0.000	6.4278 E-11	6.4113 E-11
0.125	4.7378 E-10	1.4645 E-10
0.250	5.2047 E-09	1.9111 E-09
0.375	1.5281 E-08	5.6158 E-09
0.500	2.4509 E-08	8.8518 E-09
0.625	2.5265 E-08	9.1373 E-09
0.750	1.5563 E-08	5.6666 E-09
0.875	3.2941 E-09	1.0112 E-09
1.000	5.6254 E-11	5.6239 E-11

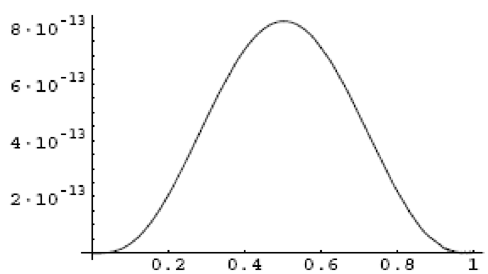

Fig. 1: Absolute error between exact and approximate solution (n = 30)

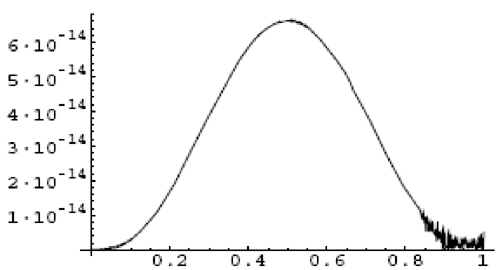

Fig. 2: Absolute error between exact and approximate solution (n = 50)

NUMERICAL EXAMPLES

All the numerical computations performed using Mathematica version 5.2.

Example 1: Consider the following nonlinear seventh order boundary value problem Siddiqi *et al*. (2012a):

$$\left.\begin{aligned} u^{(7)}(x) &= -e^{x}\,u^{2}(x),\\ u(0) &= u^{(2)}(0)=1,\ u^{(1)}(0) = u^{(3)}(0)=-1,\\ u(1) &= u^{(2)}(1)=e^{-1},\ u^{(3)}(1)=-e^{-1} \end{aligned}\right\} \tag{15}$$

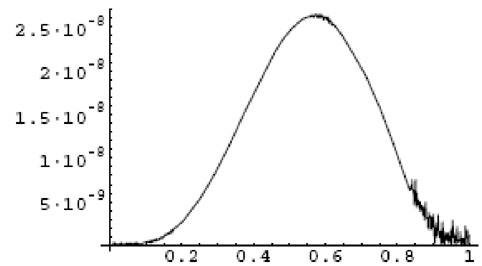

Fig. 3: Absolute error between exact and approximate solution (n = 30)

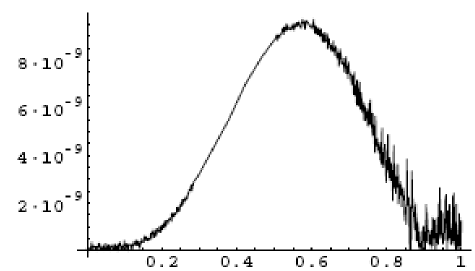

Fig. 4: Absolute error between exact and approximate solution (n = 50)

The exact solution of the Example 1 is $u(x) = e^{-x}$. Numerical results are given in Table 1 and Fig. 1 and 2.

Example 2: Consider the following nonlinear seventh order boundary value problem:

$$\left.\begin{aligned} &u^{(7)}(x)+u^{(4)}(x)-e^{u(x)}\,u(x)= e^{x}\,((-4(-3+x)+e^{(-e^{x}(x-1)\cos x)}\\ &(x-1))\cos x-8(5+x)\sin x),\\ &u(0) =1,\ u^{(1)}(0)=0=u(1),\ u^{(1)}(1) =-e\cos 1,\\ &u^{(2)}(0) =-2=u^{(3)}(0),\ u^{(2)}(1)=-2e\cos 1+2e\sin 1 \end{aligned}\right\} \tag{16}$$

The exact solution is $u(x) = e^{x}(1-x)\cos x$. The comparisons of the errors in absolute solution obtained from the present method and exact solution is shown in Table 2 and Fig. 3 and 4.

CONCLUSION

An iterative method is used to find the approximate solution of the nonlinear seventh order boundary value problem in the reproducing kernel space. In this method, an iterative sequence is obtained which is proved to converge to the exact solution uniformly. Numerical results show that the method used in the paper is valid. Compared with other method, the results of numerical example demonstrate that the present method is more accurate than existing method developed by Siddiqi *et al.* (2012a). It is worthy to note that the present method can be used as a very accurate algorithm for solving nonlinear seventh order boundary value problems.

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