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Research Article

Construction of Measurable Incidence and Adjacency Matrices from Product Measures

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Abstract: This study presents a new method of representing graphs and a new approach of constructing both incidence and adjacency matrices using the theory of product measures. It further shows that matrices constructed by this approach are measurable which a major advantage of this method is.

Keywords: Adjacency matrix, incidence matrix, product measures, measures

INTRODUCTION

In Computer Science and Mathematics, a graph is defined as a mathematical structure that is normally used to model a pairwise relation between elements from some collection. But, in the most common sense of the word, a graph can be defined as an ordered pair $G = (V, E)$ where the set V consists of the vertices or nodes while the set E is made up of the edges or lines which connect the nodes (Balakrishnan and Ranganathan, 2000). There are various types of graphs such as bipartite graphs, hyper-graphs, directed graphs (digraphs), multiline graphs, networks, line graphs, planar graphs, undirected graphs, vertex-transitive graphs, etc (Balakrishnan and Ranganathan, 2000; Chen and Hwang, 2000; Diestel, 2006; Li *et al*., 2011; Mandl, 1979; Schrijver, 2009; Strayer, 1992; Xu, 2003, 2001); and it is clear that their construction resulted from everyday life problems and hence the need to research about them.

Graphs are normally represented in two forms namely; Adjacency matrix form and Incidence matrix form and it is in either of these two forms that graphs are commonly stored in computers (Xu, 2003). Therefore, all other types of matrices namely: Toeplitz matrices, Circulant matrices, Null matrices, Triangular matrices, Diagonal matrices, etc (Agaian, 1985; Bhatia, 2007; Brauer and Gentry, 1968; Fiedler, 1971) are either adjacency matrices or incidence matrices in origin depending on the constraints subjected to the graph in question.

In this study, we would represent a graph as $G = ((V,\ A, \mu)),\ (E,\ \beta,\ v$^ where V is the vertex set of G, A is the sigma-algebra of V and μ is a measure defined on A while E is the edge set of G, β is the sigma-algebra of E and v is a measure defined on β. The goal of this study is to show that the product measure spaces ($V{\times}E$, $A{\otimes}\beta$, $\mu{\times}v$) and $V{\times}V$, $A{\otimes}A$, $\mu{\times}\mu$) represent the incidence and adjacency matrices respectively and that they are measurable (Berberian, 1999; Friedman, 1982; Hewitt and Stromberg, 1965; Lieb and Loss, 2001).

This study presents a new method of graph representation. It also proposes a new approach of constructing both incidence and adjacency matrices based on the principles of product measures; and demonstrate that both the proposed graph representation and the proposed construction algorithm possess the characteristic or feature of measurability, which is a major advantage of both methods. Finally, some areas where the research can be found applicable are discussed.

Crucial definitions and theorems: In this section, we will define the important terms and prove the theorems that would be used in the construction of the said matrices from the theory of product measures. We start by fixing some notations. Let (V, A, μ) and (E, β, v) be measurable spaces. A subset $\Psi \subseteq V{\times}E$ is called a measurable rectangle if $\Psi = \Omega \times \phi$ for some $\Omega \in A$ and $\phi \in \beta$. We let R to denote the class of all measurable rectangles with the fact in mind that, it is not a σ-algebra but a semi-algebra of the subsets of $V{\times}E$. The sigma-algebra of the subsets of $V{\times}E$ generated by the semialgebra R is called the product sigma-algebra and is denoted by A⊗B.

Definition 1: Let (X, A_1 μ_1 be a measure space and let a mapping ϕ exists such that:

$$0: X \rightarrow R^*$$

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where, R* denotes the set of extended real numbers. Then the mapping ϕ is said to be measurable if f ϕ^{-1} (I) $\in A_1$ for all intervals $I \subseteq A_1$ or if ϕ^{-1} ([c,+∞)) $\in A_1$ for all $c \in$ e R.

Definition 2: Given a set a power set Ψ a power of Ψ (A) and μ a measure defined on A; the triple (Ψ, A, μ) is called a measure space. If μ (Ψ) <∞ then (Ψ, A, μ) is called a finite measure space otherwise it is a non-finite measure space.

Theorem 1: Given two measure spaces (X, A_1, μ_1) and (Y, A_2, μ_2) together with projection maps of the form:

$$\phi_X: X \times Y \rightarrow X$$
$$\phi_Y: X \times Y \rightarrow Y$$

then the following statements hold:

- ϕ_x is a $A_1 \otimes A_2$ measurable map
- ϕ_Y is a $A_1 \otimes A_2$ measurable map

Proof (i): Let the projection maps $\phi_X: X \times Y \rightarrow X\,(\phi_Y: X \times Y)$ be defined as:

$$\phi_X(x, y) = x \quad \text{and} \quad (\phi_Y(x, y) = y$$

$\forall x < \in E\ X$ and $\forall y \in Y$. We consider the map ϕ_X and make the following analysis:

$$X \times Y \rightarrow X$$
$$A_1 \otimes A_2 \rightarrow A_1$$
$$\phi_X:\ X \times Y \rightarrow X$$
$$\phi_X:\ (x, y) = x$$

$\forall$ (x, y) $\in X \times Y$. We now claim that ϕ_X is $A_1 \otimes A_2$ measurable and hence $\forall A \in A_1$ we would show that the inverse projection map $\phi_X^{-1}(A)$ is also in $A_1 \otimes A_2$ as follows:

$$\phi_X^{-1}(A) = \{(x, y \in X \times Y | x \in A\} = A \times Y \in R \subseteq A_1 \otimes A_2$$

We have seen from above that for any set $A \in A_1$ the inverse projection map $\phi_X^{-1}(A) \in A_1 \otimes A_2$ which implies that the map ϕ_X is a $A_1 \otimes A_2$ measurable map.

- Here we also claim that ϕ_Y is $A_1 \otimes A_2$ measurable and consider the projection map:

$$\phi_y: X \times Y \longrightarrow Y$$

which is defined as:

$$\phi_y(x, y) = y$$

$\forall$ (x, y) $\in X \times Y$. Now we would show that $\forall M \in A_2$ the inverse projection map $\phi_Y^{-1}(M)$ is also in sigma-algebra $A_1 \otimes A_2$ we would proceed as follows. Given any set M which is any element of the sigma-algebra A_2 then the inverse map $\phi_Y^{-1}(M)$ becomes:

$$\phi_Y^{-1}(M) = \{(x, y \in X \times Y | y \in M\} = X \times M \in R \subseteq A_1 \otimes A_2$$

Therefore, since $\forall M \in A_2$, $\phi_Y^{-1}(M) \in A_1\ g \otimes A_2$ the fact that the mapping ϕ_Y is $A_1 \in A_2$ measurable is established.

Theorem 2: Let (X, A_1, μ_1) and (Y, A_2, μ_2) be two measurable spaces such that theorem holds. Then the σ-algebra $A_1 \in A_2$ is a member of the a-algebra of the subsets of $X \times Y$.

Proof: Let Z be any σ-algebra of the sub-sets of $X \times Y$ such that ϕx and ϕy are both Z-measurable (Theorem 1). Our goal is to show that $Z \supseteq A_1 \otimes A_2$ and we proceed by supposing that the set $A \in A_1$ and the set $B \in A_2$, then from theorem 1 it is clear enough that the following equations:

$$A \times Y = \phi_X^{-1}(A) \in Z$$
$$X \times B = \phi_Y^{-1}(B) \in Z$$

are true. Since $A \times B \in R$, we can without fear evaluate the product $A \times B$ as follows:

$$A \times B = (A \times Y) \cap (X \times B) = \phi_X^{-1}(A) \cap \phi_Y^{-1}(B) \in Z$$

which implies that $R \subseteq Z$ and hence the following result:

$$A_1 \otimes A_2 = Z\,(1) \subseteq Z \qquad (1)$$

For more clarity of Eq. (1) we let X and Y to be non-empty sets and supposed C and D to be families of subsets of X and Y respectively. Thus we can analyze as follows:

$$\begin{array}{lll} X & Y & X \times Y \\ C & D & C \times D \\ Z(C) & Z(D) & Z(C \times V) \end{array}$$

and hence Eq. (1) becomes:

$$A_1 \otimes A_2 = Z(C \times D) \subseteq Z(C) \otimes Z(D).$$

Construction from product measures: In this section, we would describe the construction of both incidence and adjacency matrices using the concepts of product measures.

We let (V, A_1, μ_1) and (E, A_2, μ_2) to be spaces where V and E are sets, A_1 and A_2 are the sigma-algebras of the sets V and E respectively and μ_1 and μ_2 are measures defined on A_1 and A^2 respectively. We want to show that the spaces ($V\times E$, $A_1\otimes A_2$, $\mu_1\times\mu_2$) and ($V\times V$, $A_1\otimes A_2$, $\mu_1\times\mu_2$) represent both incidence and adjacency matrices and they are measurable. For the construction to be meaningful, we only need to show that there exists measure Q such that:

$$\Omega: A_1\otimes A^2 \rightarrow [0, +\infty]$$

such that:

$$\Omega: (A\times B) = \mu_1 (A)\, \mu_2 (B)$$

for every $A\in A_1$, $B\in A_2$. It is a clear fact that:

$$A_1\otimes A_2 = S(R)$$

(Theorem 2) where S is any sigma-algebra of the subsets of $V\times E$ and R denotes the semi-algebra of the subsets of $V\times E$ and hence the following equation is true:

$$R = \{A\times B | A \in A_1, B \in A_2\}$$

where,

$$\mu_1: A_1 \rightarrow [0, +\infty]$$
$$\mu_2: A_2 \rightarrow [0, +\infty]$$

Step I: We let:

$$\Omega: R\rightarrow [0, +\infty]$$

defined by:

$$\Omega (A\times B) = \mu_1 (A)\, \mu_2 (B)$$

For every $A\in V$ and for every $B\in E$ and show that Ω is a measure on R. We proceed as follows. It is obvious that Q satisfies the null-empty set property, i.e:

$$\Omega (\phi) = 0$$

Next, we would show that Ω is countable additive i.e:

$$\Omega (A\times B) = \sum_{t=1}^{\infty} \Omega (A_t\times B_t)$$

where, A_1, A_t e A_1, B_1, $B_t \in A_2$ and $(A_t\times B_t)\cap (A_r\times B_r) = \phi$ for $t\neq r$. We begin by letting A and B to be two pair-wise disjoint sets which means that:

$$A\times B = \coprod_{t=1}^{\infty}(A_t \times B_t)$$

We now fix $x\in A$, vary $y\in B$ and maintain the condition $(x, y) \in A\times B$. The above operation implies that there exists a t such that $(x, y) \in A_t\times B_t$ which in-turn means that $x\in A_t$ and such that $y\in B_t$. Thus $y\in B$ implies that $y\in B_t$ where $x\in A_t$ and we have:

$$\coprod_{t\in F(x)} B_t \quad (2)$$

where,

$$F(x) = \{t\in \mathbb{N} |\, x\in A_t\}$$

Equation (2) can be further written in the form

$$\mu_2 (B) = \sum_{t\in F(x)} \mu_2 (B_t) \quad (3)$$

If $x \notin A$, then $z \notin A_t\ \forall_t$ and we can write $X A_t (x) = 0$ where $\chi_{At} (x)$ is the character is- tic function of A_t with respect to x . Also, if $x \in A$ then $x \in A_t$ for all t in $F(x)$ and hence $\chi_{At} (x) = 1$. Thus Eq. (3) can be modified as follow:
Therefore,

$$\mu_2(B)\chi_A(x) = \sum_{t=1}^{\infty} \chi_{At} (x)\mu_2(B_t) \quad (4)$$

Applying the monotone convergence theorem (MCT) on (V, A_1, μ_1) Eq. (4) becomes:

$$\int \mu_2 (\boldsymbol{B})\, \chi_A(x)\, d\, \mu_1(x) = \sum_{t=1}^{\infty} \int \chi_{At} (x)\mu_2(B_t)d\mu_1(x)$$
$$\mu_2(B_t) \int \chi_A\, d\mu_1(x) = \sum_{t=1}^{\infty} \mu_2(B_t)\, \mu_1(x)$$
$$\mu_2(B)\, \mu_1(\boldsymbol{A}) = \sum_{t=1}^{\infty} \mu_2(B_t)\, \mu_1(A_t)$$
$$\Omega (A\times B) = \sum_{t=1}^{\infty} \Omega (A_t\times B_t)$$

Hence Ω is countable additive. Therefore,

$$\Omega: A_1 \times A_2 \rightarrow [0, +\infty]$$

Such that:

$$\Omega(A\times B) = \mu_1(A)\, \mu_2(B)$$

Is a measure on the semi-algebra $A_1\times A_2$

Step II: We use the general extension theory to extend Ω to a unique measure $\overline{\Omega}$ on the sigma- algebra generated by R and claim that μ_1 and μ_2 are σ-finite i.e., given:

$$\overline{\Omega}: A_1 \otimes A_2 \longrightarrow [0, +\infty]$$

such that:

$$\overline{\Omega}(A\times B) = \Omega (A\times B)$$

then $\overline{\Omega}$ is a measure. We proceed as follows.
For our claim above to be meaningful the equations below must be true:

$$V = \coprod_{i=1}^{\infty} V_i, V_i \in A_1 \quad (5)$$

and $\mu_1(V_i)<+\infty$ for all i:

$$V\times E\coprod_{j=1}^{\infty} E_i, E_j \in A_2 \tag{6}$$

and $\mu_2(E_i)<+\infty$ for all j .Thus:

$$V\times E = (\coprod_{i=1}^{\infty} V_i)\times(\coprod_{j=1}^{\infty} E_i) = \coprod i = 1 \coprod j = 1\ (Vi\times E_i)$$

is a partition of $V\times E$ by elements of R such that:

$$\Omega(Vi\times E_i) = \mu_1(Vi)\ \mu_2\ (E_i)<+\infty.$$

Therefore, Ω isσ-finite.

The above described construction goes for incidence matrices while that of adjacency matrices follows the same procedures as above.

DISCUSSION

The concept of measurability is very important in various disciplines such as, analysis, statistics, economics, computer science, etc.

In statistics related areas the presence of the power set in this representation, which is equivalent to the universal set makes it possible for the probabilities to be calculated and hence represented in a matrix form given rise to matrices like, doubly stochastic matrices, right stochastic matrices, left stochastic matrices, etc.

In the computer science disciplines, a knowledge of the sigma algebra backed by measurability makes computation of any matrix represented in this form to be faster than other matrices of different representation as the algorithm is knowledgeable about the power set.

In Graph Theory, the power set (sigma algebra) of this representation can be used to store all perfect matchings of a graph (bipartite graph) and hence doubly stochastic matrices can be generated since perfect matchings are associated with the graphs of doubly stochastic matrices. At the same time permutation matrices can be obtain because of the fact that any convex combination of permutation matrices gives rise to a doubly stochastic matrix (Ando, 1989; Bhatia, 1997; Borwein and Lewis, 2000).

CONCLUSION

We have presented a new approach of graph representation and at the same time a new method of constructing measurable incidence and adjacency matrices. The advantage of this noble work is simply measurability as both the representation and the construction are measurable.

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