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Research Article

A Novel Hybrid Solving Approach Based on Combining Similarity Solutions with Laplace Transformation Technique to Solve Different Engineering Problems

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Abstract: In this study Laplace transformation technique combined with similarity solutions are used to solve PDE involves derivatives with respect to time and two spatial parameters. The hybrid approach is based on transforming the PDE from the real physical time domain to the Laplacian domain. The obtained PDE in the Laplacian domain involves only derivatives with respect to the two spatial parameters. This transformed PDE is then solved by similarity solution approach to convert it from a PDE in the Laplacian domain to an ODE in another domain involves independent parameter consists of the Laplace parameter s and the two independent spatial parameters. A case is discussed to demonstrate the capabilities of the proposed approach in solving different engineering problems.

Keywords: Dispersionless KP, Laplace transformation, ODE, PDE, similarity solutions

INTRODUCTION

Partial differential equations are used to formulate physical problems involving functions of several variables; such as the propagation of electromagnetic waves, sound and heat, electrostatics, electrodynamics, fluid flow and elasticity. Different physical phenomena may have identical mathematical formulations and thus be governed by the same underlying dynamic, as discussed by Stanley (1993). In the literature, there are numerous numbers of analytical, semi-analytical, hybrid and numerical techniques used to solve different types of PDE under different assumptions, conditions and applications. Some of these methods were presented in Meleshko (2005). Examples of these analytical and semi-analytical solving techniques were presented in Scott (2003), Arthur (1995) and Richard and Roth (1984). These techniques include separation of variables, integral transforms, method of characteristics, change of variables, fundamental solution, superposition principles, Laplace transformation technique, Fourier series and Fourier integral technique, similarity solution technique, trial solution methods (collocation, sub domain, least square and Galerkin methods), variation methods and combining trial solution methods with Laplace transformation, trial solutions were discussed by Al-Nimr *et al*. (1994) and Kiwan *et al*. (2000).

Other methods used for non-linear PDE are the Split-step method which has been discussed in Tsuchiya *et al*. (2001), the h-principle method presented in Eliashberg and Mishachev (2002) to solve underdetermined equations. The Riquier-Janet theory, to obtain information about many analytic over determined systems, was investigated in Fritz (1984). The method of characteristics (Similarity Transformation method) used in some very special cases to solve partial differential equations, the perturbation analysis in which the solution is considered to be a correction to an equation with a known solution. Elemér (1987) presented generalized solutions of PDEs. Other methods used to solve non-linear PDE are the Continuous group theory, Lie algebras and differential geometry that are used to understand the structure of linear and nonlinear partial differential equations for generating integrable equations and the Almost-solution of PDE which is a concept introduced by a Russian mathematician Vladimir Miklyukov.

Examples of numerical methods used to solve PDEs are the Finite Element Method (FEM), Finite Volume Methods (FVM) and Finite Difference Methods (FDM). The FEM is the most efficient one among these methods and especially its exceptionally efficient higher-order version hp-FEM. FEM method was discussed in Pavel (2005), FVM was presented in Randall (2002) and FDM can be found in Mitchell and Griffiths (1997). Other versions of FEM include the Generalized Finite Element Method (GFEM), Extended Finite Element Method (XFEM), Spectral Finite Element Method (SFEM), mesh-free finite element

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method, Discontinuous Galerkin Finite Element Method (DGFEM).

The present study proposes a novel hybrid technique that combines the Laplace transformation technique (Richard and Roth, 1984) with the similarity solution approach Arthur (1995) to solve transient PDE that involves derivative with respect to time and derivatives with respect to two other spatial independent variables. Laplace transformation will transform the time dependent PDE from the time domain to the s-domain to yield a PDE that involves only derivatives with respect to the two independent spatial variables. The obtained PDE in the s-domain is then solved using similarity solution approach to convert it to an ODE in the similarity domain and the solution will be presented in terms of the s parameter and the similarity parameter that combines the two independent spatial parameters. The solution of the obtained ODE is then inverted back to the time domain either analytically (Bateman and Erd´elyi, 1954; Doetsch, 1958; Ditkin and Prudnikov, 1965) or numerically (Tzoub *et al*., 1997) and this solution represents the final solution of the considered PDE. The proposed novel method will be demonstrated by solving two dimensional time dependent case.

ANALYSIS

The two dimensional time dependent partial differential equation with a form similar to the Dispersionless Kadomtsev-Petviashvili (DKP) equation (Konopelchenko *et al*., 2001)):

$$\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial\left(\frac{u \partial u}{\partial x}\right)}{\partial x} + \lambda \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

$$u(x, y, t) \ \mathrm{x, y, t} \in R$$

The DKP equation arises in many important physical applications especially in wave modeling. The form discussed here will be linear and will not include the non-linear term $\frac{\partial\left(\frac{u \partial u}{\partial x}\right)}{\partial x}$ (i.e., negligible convection), If the surface tension is weak compared to the gravitational forces then $\lambda = +1$ and if the surface tension is stronger than gravitational forces $\lambda = -1$ (Kadomtsev and Petviashvili, 1970). Here the nonlinear term will be neglected and consider a strong surface tension forces. Then Eq. (1) reduces to:

$$\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (2)$$

Consider the following equation:

$$\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (3)$$

With the following initial and boundary conditions:

$$\frac{\partial u(x,y,0)}{\partial x} = 0 \,, u(0, y, t) = Q1 \,, u(x, 0, t) = Q2$$

Laplace transformation: Taking the Laplace transform for Eq. (3) yields:

$$L\left(\frac{\partial^2 u}{\partial x \partial t}\right) = s\frac{\partial w(x,y,s)}{\partial x} - \frac{\partial u(x,y,0)}{\partial x} = s\frac{\partial u(x,y,s)}{\partial x} \quad (4)$$

also,

$$L\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 w(x,y,s)}{\partial y 2} \quad (5)$$

and

$$\mathrm{L(Q1)} = \frac{\mathrm{Q1}}{\mathrm{s}} \ \& \ L(\mathrm{Q2}) = \frac{\mathrm{Q2}}{\mathrm{s}}$$

So Eq. (3) becomes:

$$s\frac{\partial w(x,y,s)}{\partial x} - \frac{\partial^2 w(x,y,s)}{\partial y^2} = 0 \quad (6)$$

Similarity solution: The similarity solution is then applied on Eq. (6). Assuming the following transition:

$$w = \propto^c \ w'$$
$$y = \propto^b \ y'$$
$$x = \propto^a \ x'$$

$$s \propto^{c-a} \frac{\partial w'(x,y,s)}{\partial x'} - \propto^{c-2b} \frac{\partial^2 w'(x,y,s)}{\partial y'^2} = 0 \quad (7)$$

And in order for Eq. (7) to be invariant:

c- *a* = *c*-2*b* then *a* = 2*b*

Note that:

$$\frac{w'}{x'^{\frac{c}{a}}} = \frac{\propto^{-c} w}{(\propto^{-a} x)^{\wedge c}/a} = \frac{w}{x^{\frac{c}{a}}} \quad (8)$$

also,

$$\frac{y'}{x'^{\frac{b}{a}}} = \frac{\propto^{-b} y}{(\propto^{-a} x)^{\frac{b}{a}}} = \frac{y}{x^{\frac{b}{a}}} \quad (9)$$

So both grouping of variables are invariant under the transformation, then we can assume:

$$w = x^{\frac{c}{a}} F(\dot{\eta}, S) \text{ where } \dot{\eta} = \frac{y}{x^{\frac{b}{a}}} = \frac{y}{x^{\frac{1}{2}}}$$

$$\frac{\partial w\,(x,y,s)}{\partial x} = x^{\frac{c}{a}} \frac{\partial F}{\partial \dot{\eta}} \frac{\partial \dot{\eta}}{\partial x} + \frac{c}{b}\, x^{\frac{c}{a}-1}\, \mathrm{F} \quad (10)$$

$$\text{but, } \frac{\partial \dot{\eta}}{\partial x} = -\frac{1}{2} \dot{\eta} x^{-1}$$

then $\frac{\partial w(x,y,s)}{\partial x} = \ x^{\frac{c}{a}-1}\left(-\frac{1}{2}\dot{\eta}\ \frac{\partial F}{\partial \dot{\eta}}\ \ +\frac{c}{b}\ \mathrm{F}\right)$ (11)

also,

$$\frac{\partial^2 w(x,y,s)}{\partial y^2} = \frac{\partial}{\partial y}\left(x^{\frac{c}{a}}\frac{\partial F}{\partial \dot{\eta}}\ \frac{\partial \dot{\eta}}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^{\frac{c}{a}-\frac{1}{2}}\frac{\partial F}{\partial \dot{\eta}}\right) = x^{\frac{c}{a}-1}\ \frac{\partial^2 F}{\partial \dot{\eta}^2} \quad (12)$$

So Eq. (6) becomes:

$$S\ x^{\frac{c}{a}-1}\left(-\frac{1}{2}\dot{\eta}\ \frac{\partial F}{\partial \dot{\eta}}\ \ +\frac{c}{b}\ \mathrm{F}\right) - x^{\frac{c}{a}-1}\ \frac{\partial^2 F}{\partial \dot{\eta}^2} = 0 \quad (13)$$

$$S\left(-\frac{1}{2}\dot{\eta}\ \frac{\partial F}{\partial \dot{\eta}}\ \ +\frac{c}{b}\ \mathrm{F}\right) - \frac{\partial^2 F}{\partial \dot{\eta}^2} = 0 \quad (14)$$

But now if we consider the assumed boundary conditions:

$$u(0,y,s) = x^{\frac{c}{a}}\ F(\infty,S) = \frac{Q1}{s}\ ,u(x,0,s) = x^{\frac{c}{a}}\ F(0,S) = \frac{Q2}{s}$$

Assuming that $Q1$ or $Q2 \neq 0$ then $\frac{\mathrm{c}}{\mathrm{a}} = 0$ which means that c = 0. Equation (14) will then be simplified to become as follows:

$$\frac{1}{2}\dot{\eta}\ \frac{\partial F}{\partial \dot{\eta}} + \frac{\partial^2 F}{\partial \dot{\eta}^2} = 0 \quad (15)$$

Equation (15) is an ODE in $\dot{\eta}$ and it could be solved analytically as follow:

$$\frac{1}{2}S\dot{\eta}F' + \mathrm{F}'' = 0 \quad (16)$$

Then,

$$\frac{F''}{F'} = -\frac{1}{2}S\dot{\eta} \quad (17)$$

Integrating both sides, we get:

$$F' = C1e^{-\frac{S\dot{\eta}^2}{4}} \quad (18)$$

$$\int F'd\dot{\eta} = \int C1e^{-\frac{S\dot{\eta}^2}{4}}\ d\dot{\eta} \quad (19)$$

$$F(\dot{\eta},S) = C3\,\mathrm{erf}\left(\sqrt{\frac{\dot{\eta}^2 s}{4}}\right) + C2 \quad (20)$$

This Function of S and $\dot{\eta}$ could be transformed to the time domain using Laplace inversion, either analytically or using a computer program, here Eq. (15) will be transformed for special boundary conditions which imply special values of the constants C3 and C2 take:

$$(0,y,s) = \frac{Q1}{s}\ ,u(x,0,s) = \frac{Q2}{s}$$

$$uF(\infty,S) = \frac{Q1}{s}\ ,F(0,S) = \frac{Q2}{s}$$

$$F(\infty,S) = \frac{Q1}{s} = \ C3\,\mathrm{erf}(\infty)\ + C2$$

with $\mathrm{erf}(\infty) = 1$ then,

$$C3\ + C2 = \frac{Q1}{s} \quad (21)$$

Also:

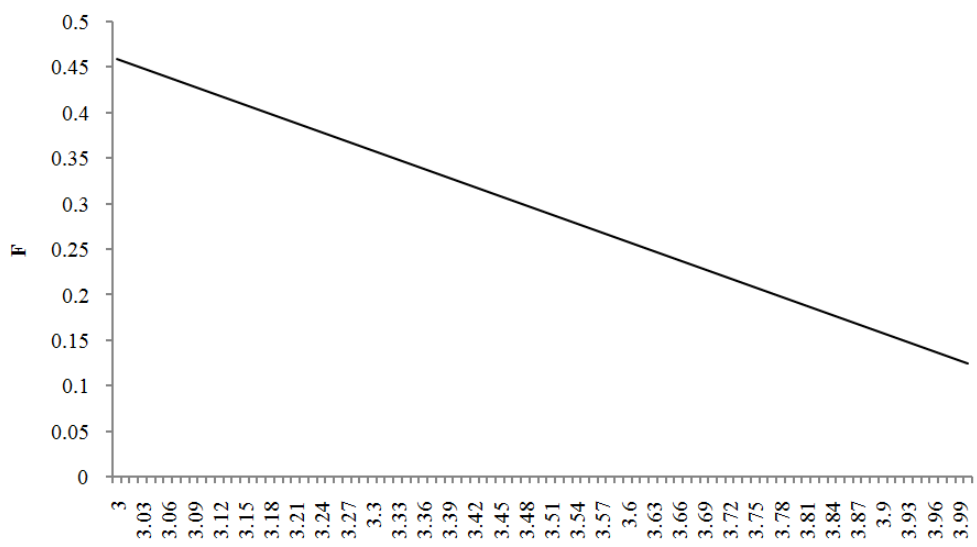

Fig. 1: The value of function F at (real time = 4. s) and different values for (n)

$$F(0,S) = \frac{Q2}{s} = C3\,\mathrm{erf}(0) + C2$$

with erf $(0) = 0$ then,

$$C2 = \frac{Q2}{s} \text{ and } C3 = \frac{(Q1-Q2)}{s}$$

Therefore, Eq. (6) becomes:

$$F(\dot{\eta}, S) = s^{-1}\left((Q1 - Q2)\,\mathrm{erf}\left(\sqrt{\frac{\dot{\eta}^2 s}{4}}\right) + Q2\right) \quad (22)$$

And for the special values of Q1 = 0 and Q2 = 1, Eq. (22) reduces to:

$$F(\dot{\eta}, S) = \frac{1-\mathrm{erf}\left(\sqrt{\frac{\dot{\eta}^2 s}{4}}\right)}{s} = \frac{erfc\left(\sqrt{\frac{\dot{\eta}^2 s}{4}}\right)}{s} \quad (23)$$

The Laplace inversion of the right side of Eq. (23) could be obtained using a computer program based on Riemann sum (Tzoub *et al*., 1997), here the special values of Q1 = 0 and Q2 = 1 are considered and the Laplace inversion is performed at a certain value of real time (t = 4) within an interval $3<\dot{\eta}<4$. Using a proper error function expansion (Prudnikov *et al*., 1990), Fig. 1 is created. The obtained Points were fitted by the straight line as shown in Fig. 1.

CONCLUSION

In this study a new hybrid method for solving time dependent PDEs were developed. This hybrid method combines the Laplace transformation with the similarity solution technique by transforming the PDE from the time domain to the s-domain and then applying the similarity solution which will yield an ODE that could be solved by various methods. The ODE solution is then inverted to the time domain.

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