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Research Article

A Note on Abel-Grassmann's Groupoids

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Abstract: In this study we have constructed various AG-groupoids from vector spaces over finite fields and also from finite fields by defining new operations on these structures.

Keywords: AG-groupoid, Cayley diagram, Galois field

INTRODUCTION

An Abel-Grassmann's groupoid (Protić and Stevanović, 2004), abbreviated as an AG-groupoid, is a groupoid S whose elements satisfy the invertive law:

$$(ab)c = (cb)a, \text{ for all } a, b, c \in S. \quad (1)$$

It is also called a left almost semigroup (Kazim and Naseeruddin, 1972; Mushtaq and Iqbal, 1990). In Holgate (1992) it is called a left invertive groupoid. In this study we shall call it an AG-groupoid. It has been shown in Mushtaq and Yusuf (1978) that if an AG-groupoid contains a left identity then the left identity is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup.

An AG-groupoid S is medial (Kazim and Naseeruddin, 1972) that is:

$$(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S. \quad (2)$$

An AG-groupoid is called an AG-band if all its elements are idempotents.

A commutative inverse semigroup (S, *) becomes an AG-groupoid (S,.) under the relation $a \cdot b = b * a^{-1}$ (Mushtaq and Yusuf, 1988).

In Stevanović and Protić (2004) a binary operation "$\circ$" on an AG-groupoid S has been defined as follows: for all $x, y \in S$ there exist a such that $x \circ y = (xa)y$. Clearly $x \circ y = y \circ x$ for all $x, y \in S$.

Now if an AG-groupoid S contains a left identity e then the operation $\circ$ becomes associative, because using (1) and (2), we get:

$$(x \circ y) \circ z = (((xa)y)a)z = (za)((xa)y) = (e(za))((xa)y)$$
$$= (xa)((za)y) = (xa)((ya)z) = x \circ (y \circ z).$$

Hence (S, $\circ$) is a commutative semi group. Connection discussed above make this non-associative structure interesting and useful.

PRELIMINARIES

Here we construct AG-groupoids by defining new operations on vector spaces over finite fields. AG-groupoids constructed from finite fields are very interesting. It is well known that a multiplicative group of a finite field is a cyclic group generated by a single element. By using these generators we have drawn the Cayley diagrams for such AG-groupoids which have been constructed from finite fields. The diagrams are either bi-partite (that is, their vertices can be colored by using two minimum colors) or tri-partite (that is, they can be colored using three minimum colors).

Here we begin with the examples of AG-groupoids having n o left identity.

Example 1: Let $S = \{1,2,3,4,5,6,7\}$, the binary operation . be defined on S as follows:

.	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	6	3	7	4	1	5	2
3	4	1	5	2	6	3	7
4	2	6	3	7	4	1	5
5	7	4	1	5	2	6	3
6	5	2	6	3	7	4	1
7	3	7	4	1	5	2	6

Then $(S, \cdot)$ is an AG-groupoid without left identity.

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Following is an example of an AG-groupoid with the left identity.

Example 2: Let $S = \{1,2,3,4,5,6,7,8\}$, the binary operation be defined on *S* as follows:

·	1	2	3	4	5	6	7	8
1	7	8	1	2	3	4	5	6
2	6	7	8	1	2	3	4	5
3	5	6	7	8	1	2	3	4
4	4	5	6	7	8	1	2	3
5	3	4	5	6	7	8	1	2
6	2	3	4	5	6	7	8	1
7	1	2	3	4	5	6	7	8
8	8	1	2	3	4	5	6	7

Then $(S,\cdot)$ is an AG-groupoid with left identity 7.

A graph *G* is a finite non-empty set of objects called vertices (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of *G* called edges. The vertex set is denoted by *V* (*G*), while the edge set is denoted by *E* (*G*).

A graph *G* is connected if every two of its vertices are connected. A graph *G* that is not connected is disconnected. A graph is planar if it can be embedded in the plane.

A directed graph or digraph *D* is a finite non-empty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of *D* called arcs or directed edges.

A graph *G* is n -partite, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, ..., V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j , $i \neq j$. For $n = 2$, such graphs are called bi-partite graphs.

Theorem 1: Let *W* be a sub-space of a vector space *V* over a field *F* of cardinal 2*r* such that r>1. Define the binary operation $\circ$ on *W* as follows:

$u \circ v = \alpha^r u + \alpha v$, where α is a generator of $F \setminus \{0\}$ and $u, v \in W$. Then $(W, \circ)$ is an AG-groupoid.

Proof: Clearly W is closed. Next we will show that W satisfies left invertive law:

$$(x \circ y) \circ z = \alpha^r(\alpha^r x + \alpha y) + \alpha z = \alpha^{2r} x + \alpha^{r+1} y + \alpha z = \alpha x + \alpha^{r+1} y + \alpha z. \quad (3)$$

Now:

$$(z \circ y) \circ x = \alpha^r(\alpha^r z + \alpha y) + \alpha x = \alpha^{2r} z + \alpha^{r+1} y + \alpha x = \alpha z + \alpha^{r+1} y + \alpha x = \alpha x + \alpha^{r+1} y + \alpha z. \quad (4)$$

From (3) and (4), we get:

$$(x \circ y) \circ z = (z \circ y) \circ x, \text{for all } x, y, z \in W.$$

Hence $(W, \circ)$ is an AG-groupoid.

It is not a semigroup because:

$$x \circ (y \circ z) = \alpha^r x + \alpha(\alpha^r y + \alpha z) = \alpha^r x + \alpha^{r+1} y + \alpha^2 z. \quad (5)$$

(3) and (5) imply that:

$$(x \circ y) \circ z \neq x \circ (y \circ z), \text{for some } x, y, z \in W.$$

Also $(W, \circ)$ is not commutative because:

$u \circ v = \alpha^r u + \alpha v$, and

$v \circ u = \alpha^r v + \alpha u$, so

$u \circ v \neq v \circ u$, for some $u, v \in W$.

Hence $(W, \circ)$ is an AG-groupoid.

Remark 1: An AG-groupoid $(W, \circ)$ is referred to as an AG-groupoid defined by the vector space $(V, \cdot, +)$.

Remark 2: If we take $u, v \in F$, taking α as a generator of *F* and cardinal of *F* is 2*r*, then $(F, \circ)$ is said to be an AG-groupoid defined by Galois field.

An element a of an AG-groupoid S is called an idempotent if and only if $a = a^2$.

An AG-groupoid is called AG-band if all its elements are idempotents.

CAYLEY DIAGRAMS

A Cayley graph (also known as a Cayley colour graph and named after A. Cayley), is a graph that encodes the structure of a group.

Specifically, let $G = \langle X \mid R \rangle$ be a presentation of the finitely generated group G with generators *X* and relations *R*. We define the Cayley graph $\Gamma = \Gamma(G, X)$ of *G* with generators *X* as:

$$\Gamma = (G, E)$$

where,

$$E = \{\{u, a \cdot u\} \mid u \in G, a \in X\} \text{ (} E \text{ is the set of edges).}$$

That is, the vertices of the Cayley graph are precisely the elements of *G* and two elements of *G* are connected by an edge if and only if some generator in *X* transfers the one to the other. He has proposed the use of colors to distinguish the edges associated with different generators.

Remark 3: If we put the value of $r = 2$, in remark 2, we get Galois field of order 4.

Further we need to construct a field of 4 elements, for this take an irreducible polynomial $x^2 + x + 1$ in $Z_2 = \{0,1\}$. Then simple calculations yield that $GF(2^2) = \{0,1,t,t^2\}$. The table of this field is given by:

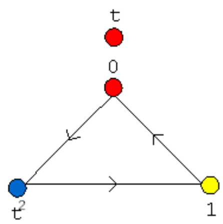

Fig. 1: Tri-pertite and planar graph

.	0	1	t	t^2
0	0	0	0	0
1	0	1	t	t^2
t	0	t	t^2	1
t^2	0	t^2	1	t

+	0	1	t	t^2
0	0	1	t	t^2
1	1	0	t^2	t
t	t	t^2	0	1
t^2	t^2	t	1	0

Example 3: Using $GF(2^2)\setminus\{0\}=F\setminus\{0\}=\langle t\ :\ t^3=1\rangle=\{1,t,t^2\}$ and $u\circ v=\alpha^2u+\alpha v$, for all $u,v\in F$ and $\alpha=t\in F$, we get the following table of an AG-groupoid:

*	0	1	t	t^2
0	0	t	t^2	1
1	t^2	1	0	t
t	1	t^2	t	0
t^2	t	0	1	t^2

We can draw the Cayley diagram for it as under, which is a tri-partite, planar disconnected graph (Fig. 1).

Theorem 2: Let W be a sub-space of a vector space V over a field F of cardinal p^n for some prime $p\neq 2$. Define the binary operation $\otimes$ on W as follows:

$u\otimes v=\alpha^{\frac{p^n-1}{2}}u+v$, where α is a generator of $F\setminus\{0\}$ and $u,v\in W$. Then $(W,\otimes)$ is an AG-groupoid with left identity 0.

Proof: Clearly W is closed. Next we will show that W satisfies left invertive law:

$$(x\otimes y)\otimes z=\alpha^{\frac{p^n-1}{2}}\left(\alpha^{\frac{p^n-1}{2}}x+y\right)+z= \quad (6)$$
$$\alpha^{p^n-1}x+\alpha^{\frac{p^n-1}{2}}y+z=x+\alpha^{\frac{p^n-1}{2}}y+z.$$

Now:

$$(z\otimes y)\otimes x=\alpha^{\frac{p^n-1}{2}}\left(\alpha^{\frac{p^n-1}{2}}z+y\right)$$
$$+x=\alpha^{p^n-1}z+\alpha^{\frac{p^n-1}{2}}y+x \quad (7)$$
$$=z+\alpha^{\frac{p^n-1}{2}}y+x$$
$$=x+\alpha^{\frac{p^n-1}{2}}y+z.$$

From (6) and (7), we get:

$$(x\otimes y)\otimes z=(z\otimes y)\otimes x,\ \text{for all}\ x,y,z\in W.$$

Hence $(W,\otimes)$ is an AG-groupoid.
It is not a semigroup because:

$$x\otimes(y\otimes z)=\alpha^{\frac{p^n-1}{2}}x+(\alpha^{\frac{p^n-1}{2}}y+z) \quad (8)$$
$$=\alpha^{\frac{p^n-1}{2}}x+\alpha^{\frac{p^n-1}{2}}y+z.$$

(6) and (8) simply that:

$$(x\otimes y)\otimes z\neq x\otimes(y\otimes z),\ \text{for some}\ x,y,z\in W.$$

Also $(W,\otimes)$ is not commutative because:

$$u\otimes v=\alpha^{\frac{p^n-1}{2}}u+v,\ \text{and}\ v\otimes u=\alpha^{\frac{p^n-1}{2}}v+u,$$
$$\text{so}\ u\otimes v\neq v\otimes u,\ \text{for some}\ u,v\in W$$

Now:

$$0\otimes x=\alpha^{\frac{p^n-1}{2}}0+x=x,\ \text{for all}\ x\in W.$$

Hence $(W,\otimes)$ is an AG-groupoid with left identity 0

Example 4: Put p = 3 and $n=1$, in theorem 2, then the cardinal of F is 3 and $u\otimes v=\alpha u+v$, for all u,v and fixed element α of F.

Obviously F = Z_3 = {0, 1, 2} mod 3, $F\setminus\{0\}=\{1,2\}=\langle 2\ :\ 2^2=1\rangle$, here $\alpha=2$, we get the following table of an AG-groupoid $\{0,1,2\}$:

⊗	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Now we can draw the Cayley diagram for the formed example of an AG-groupoid $(F,\otimes)$, which is a bi-partite, planar disconnected graph (Fig. 2).

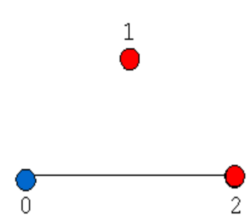

Fig. 2: Bi-partite, disconnected graph

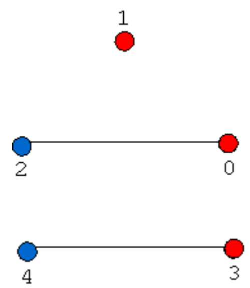

Fig. 3: Bi-partite, planar graph

Example 5: Put $p=5$ and $n=1$, in theorem 2, then we get $|F| = 5$ and $u \otimes v = \alpha^2 u + v$.

Now clearly $GF(5) = F = Z_5 = \{0,1,2,3,4\}$ mod 5, $F \setminus \{0\} = \langle 2 : 2^4 = 1\rangle$, taking α as a generator which is 2, in this case, then:

$$u \otimes v = 2^2 \cdot u + v = 4 \cdot u + v.$$

Hence we get the following AG-groupoid:

⊗	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

The Cayley diagram for the above example is given by, which is a bi-partite, planar disconnected graph (Fig. 3).

Theorem 3: Let W be a sub-space of a vector space V over a field F of cardinal r such that r>1. Define the binary operation $*$ on W as follows:

$u * v = \alpha u + \alpha^2 v$, where α is a generator of $F \setminus \{0\}$ and $u, v \in W$. Then $(W, *)$ is an AG-groupoid.

Proof: Clearly W is closed. Next we will show that W satisfies the left invertive law:

$$(x*y)*z = \alpha(\alpha x + \alpha^2 y) + \alpha^2 z = \alpha^2 x + \alpha^3 y + \alpha^2 z. \quad (9)$$

Now:

$$\begin{aligned}(z*y)*x &= \alpha(\alpha z + \alpha^2 y) + \alpha^2 x \\ &= \alpha^2 z + \alpha^3 y + \alpha^2 x = \alpha^2 x + \alpha^3 y + \alpha^2 z.\end{aligned} \quad (10)$$

From (9) and (10), we get:

$$S\,(x*y)*z = (z*y)*x, \text{for all } x, y, z \in W.$$

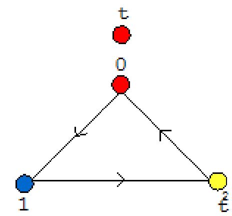

Fig. 4: Tri-partite directed graph

Hence $(W, *)$ is an AG-groupoid.
It is not a semigroup because:

$$\begin{aligned}x*(y*z) &= \alpha x + \alpha^2(\alpha y + \alpha^2 z) \\ &= \alpha x + \alpha^3 y + \alpha^4 z\end{aligned} \quad (11)$$

(9) and (11) imply that:

$$(x*y)*z \neq x*(y*z), \text{for some } x, y, z \in W.$$

Also it is not commutative because:

$$u*v = \alpha u + \alpha^2 v, \text{and } v*u = \alpha v + \alpha^2 u, \text{so}$$
$$u*v \neq v*u, \text{for some } u, v \in W.$$

Hence $(W, *)$ is an AG-groupoid.

Example 6: Let $|F| = 4$.

Obviously the field of order 4, is $GF(2^2) \setminus \{0\} = \langle t : t^3 = 1\rangle = \{1, t, t^2\}$, further put $\alpha = t$ in $u * v = \alpha u + \alpha^2 v$, for all $u, v \in F$, thus obtain the following table for an AG-band $\{0, 1, t, t^2\}$:

$*$	0	1	t	t^2
0	0	t^2	1	t
1	t	1	t^2	0
t	t^2	0	t	1
t^2	1	t	0	t^2

This table now evolves the following diagram (Fig. 4).

Example 7: Let $S = \{1, 2, 3, 4\}$, the binary operation . be defined on S as follows:

.	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

Then $(S, \cdot)$ is an AG-band, (also given in (Protić and Stevanović, 2004)). This example is a particular form of the theorem 3.

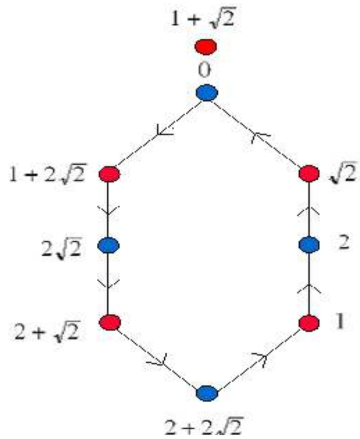

Fig. 5: Bi-partite, disconnected planar graph

Example 8: Let us put the value of $r = 9$ in theorem 3, then $|F| = 9$.

Now we need to construct a field of 9 elements, for this take an irreducible polynomial $t^2 + t + 2 + 0$ in $Z_3 = \{0, 1, 2\}$. Then simple calculations yields:

$$GF(3^2)\setminus\{0\} = F\setminus\{0\} = \langle 1+\sqrt{2} = \alpha \; : \; \alpha^8 = 1\rangle$$
$$= \{1, 2, \sqrt{2}, 2\sqrt{2}, 2+\sqrt{2}, 2+2\sqrt{2}, 1+2\sqrt{2}, 1+\sqrt{2}\}.$$

Now put the value of $\alpha = 1+\sqrt{2}$ in $u * v = \alpha u + \alpha^2 v$, we get:

$$u * v = (1+\sqrt{2})u + (1+\sqrt{2})^2$$
$$v = (1+\sqrt{2})u + 2\sqrt{2}v, \text{ for all } u, v \in F. \qquad (12)$$

Putting all the values of u, v from F in Eq. (12) we get an AG-band:

We get the following bi-partite, disconnected, planar directed graph (Fig. 5).

Remark 4: If we take finite fields instead of subspaces W of vector spaces V, in theorems 1, 2 and 3, then we can make the Cayley diagrams for these AG-groupoids by using the definition of a Cayley graph.

Example 9: Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the binary operation . be defined on S as follows:

·	1	2	3	4	5	6	7	8
1	1	2	4	4	4	4	4	8
2	8	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
5	4	4	4	4	4	4	4	4
6	4	4	4	4	4	4	4	4
7	4	4	4	4	4	4	4	4
8	2	4	4	4	4	4	4	4

It is non-commutative and non-associative because $8 = 1 \cdot 8 \neq 8 \cdot 1 = 2,\ \ 2 = (2 \cdot 1) \cdot 1 \neq 2 \cdot (1 \cdot 1) = 8$.

Also it is easy to verify that left invertive law holds in S. Hence $(S, \cdot)$ is an AG-groupoid.

Example 10: Let $S = \{1, 2, 3, 4\}$, the binary operation . be defined on S as follows:

.	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	2	3	3	3

It is non-commutative and non-associative because, $4 = 1 \cdot 4 \neq 4 \cdot 1 = 2$ and $2 = (2 \cdot 1) \cdot 1 \neq 2 \cdot (1 \cdot 1) = 4$. Thus $(S, \cdot)$ is an AG-groupoid with left identity 1.

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