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Research Article

A Univariate Symmetric C^5 Subdivision Scheme

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Abstract: Subdivision schemes play a vital role in Computer Aided Geometric Design these days. A new univariate symmetric ternary 6-point approximating subdivision scheme has been introduced that generates the limiting curve of C^5 continuity and its limit functions has a support on (-9, 8). The Laurent polynomial method has been used to investigate the continuity of the subdivision scheme. To determine the maximum degree of smoothness of the subdivision scheme, Holder exponent of the scheme has been calculated. The behavior of the proposed subdivision scheme has been depicted through four examples.

Keywords: Approximating subdivision scheme, convergence and smoothness, Laurent polynomial, mask, ternary

INTRODUCTION

During the last decade, subdivision schemes became one of the most popular tools to create smooth limiting curves and surfaces from discrete set of data points. Now subdivision has become an independent subject due to many applications in the different fields including Computer Aided Geometric Design, Computer Graphics, Graphic Animation, Image Processing, Engineering and Reverse Engineering etc.

Chaikin (1974) introduced the recursively corner cutting binary 2-point approximating subdivision scheme that generates a smooth limiting curve of C^1 continuity. Siddiqi and Rehan (2010a) developed a new corner cutting ternary 2-point approximating subdivision scheme that generates limiting curve of C^1 continuity. Hassan and Dodgson (2003) introduced a ternary 3-point approximating subdivision scheme that generates the limiting curve of C^2 continuity and also ternary 3-point interpolating subdivision scheme that generates limiting curves of C^1 continuity. Siddiqi and Rehan (2009) introduced another ternary 3-point approximating subdivision scheme that generates limiting curve of C^2 continuity. Siddiqi and Rehan (2010b) modified the ternary 3-point approximating subdivision scheme that generates family of C^1 and C^2 limiting curves. Zheng *et al*. (2007) analyzed the fractal range of a ternary 3-pointinterpolating subdivision scheme introduced by Hassan and Dodgson (2003). Zheng *et al*. (2005) also presented a ternary 3-point interpolating subdivision scheme that generates C^1 limiting curves.

Hassan *et al*. (2002) developed a ternary 4-point interpolating subdivision scheme that generates the limiting curves of C^2 continuity for certain range of tension parameter. Ko *et al*. (2007) introduced a ternary 4-point approximating subdivision scheme that generates a limiting curve of C^2 continuity. Siddiqi and Rehan (2010c) proposed another ternary 4-point approximating subdivision scheme that generates the limiting curve of C^3 continuity. Siddiqi and Rehan (2009) presented a ternary 5-point approximating subdivision scheme that generates C^4 limiting curve and its limiting functions has a support on (-5, 4).

A subdivision algorithm recursively refines the initial polygon to produce a sequence of finer polygons that converge to a smooth limiting curve. Each subdivision scheme is associated with a mask $a=\{a_i\}, i \in Z$. The ternary subdivision scheme is the process which recursively define a sequence of control points $f^k = f_i^k, i \in Z$, by the rule of the form with mask $a=\{a_i\}, i \in Z$:

$$f_i^{k+1} = \sum a_{i-3j} f_j^k, i \in Z$$

which is formally denoted by:

$$f^{k+1} = Sf^k = S^k f^0$$

A subdivision scheme is said to be uniformly convergent if for every initial data $f^0 = \{f_i\}, i \in Z$, there is a continuous function f such that for any closed interval [a, b]:

$$\lim_{k\to\infty} \sup_{i \in Z \cap 3^k[a,b]} \left| f_i^k - f(3^{-k} i) \right| = 0$$

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Obviously $f = S^{\infty} f^0$ is considered to be a limit function of subdivision scheme S.

A new ternary 6-point approximating subdivision scheme has been introduced using quintic B-spline basis functions. The ternary 6-point approximating subdivision scheme is defined as:

$$f_{3i}^{k+1} = a_{-7} f_{i-2}^{k} + a_{-4} f_{i-1}^{k} + a_{-1} f_{i}^{k} + a_{2} f_{i+1}^{k} + a_{5} f_{i+2}^{k} + a_{8} f_{i+3}^{k},$$
$$f_{3i+1}^{k+1} = a_{-8} f_{i-2}^{k} + a_{-5} f_{i-1}^{k} + a_{-2} f_{i}^{k} + a_{1} f_{i+1}^{k} + a_{4} f_{i+2}^{k} + a_{7} f_{i+3}^{k},$$
$$f_{3i+2}^{k+1} = a_{-9} f_{i-2}^{k} + a_{-6} f_{i-1}^{k} + a_{-3} f_{i}^{k} + a_{0} f_{i+1}^{k} + a_{3} f_{i+2}^{k} + a_{6} f_{i+3}^{k}$$

where, f_i^0 is a set of initial control points with the mask:

$$a_{-9} = a_{-8} = a_{-7} = \frac{1}{120}(1-w)^5,$$
$$a_{-6} = a_{-5} = a_{-4} = \frac{1}{120}(26 - 50w + 20w^2 + 20w^3 - 20w^4 + 5w^5),$$
$$a_{-3} = a_{-2} = a_{-1} = \frac{1}{60}(33 - 30w^2 + 15w^4 - 5w^5),$$
$$a_0 = a_1 = a_2 = \frac{1}{60}(13 + 25w + 10w^2 - 10w^3 - 10w^4 + 5w^5),$$
$$a_3 = a_4 = a_5 = \frac{1}{120}(1 + 5w + 10w^2 + 10w^3 + 5w^4 - 5w^5)$$

and

$$a_6 = a_7 = a_8 = \frac{1}{120}(w)^5$$

The refinement rules of the subdivision scheme have been calculated for $w = \frac{1}{6}, w = \frac{3}{6}\ and\ w = \frac{5}{6}$ as:

$$\left.\begin{aligned} f_{3i}^{k+1}\left(\frac{1}{6}\right) &= \frac{625}{186624} f_{i-2}^{k} + \frac{142301}{933120} f_{i-1}^{k} + \frac{250213}{466560} f_{i}^{k} + \frac{135233}{466560} f_{i+1}^{k} + \frac{16801}{933120} f_{i+2}^{k} + \frac{1}{933120} f_{i+3}^{k}, \\ f_{3i+1}^{k+1}\left(\frac{3}{6}\right) &= \frac{1}{3840} f_{i-2}^{k} + \frac{79}{1280} f_{i-1}^{k} + \frac{841}{1920} f_{i}^{k} + \frac{841}{1920} f_{i+1}^{k} + \frac{79}{1280} f_{i+2}^{k} + \frac{1}{3840} f_{i+3}^{k}, \\ f_{3i+2}^{k+1}\left(\frac{5}{6}\right) &= \frac{1}{933120} f_{i-2}^{k} + \frac{16801}{933120} f_{i-1}^{k} + \frac{135233}{466560} f_{i}^{k} + \frac{250213}{466560} f_{i+1}^{k} + \frac{142301}{933120} f_{i+2}^{k} + \frac{625}{186624} f_{i+3}^{k}. \end{aligned}\right\} \quad (1)$$

The proposed subdivision scheme generates the limiting curve of C^5 continuity with the wider range of support on [-9, 8] of basis functions which is helpful for the geometric designers.

ANALYSIS OF TERNARY SUBDIVISION SCHEME

For the convergent subdivision scheme S, the corresponding mask $\{a_i\}, i \in Z$ necessarily satisfies:

$$\sum_{j \in Z} a_{3j} = \sum_{j \in Z} a_{3j+1} = \sum_{j \in Z} a_{3j+2} = 1 \quad (2)$$

Introducing a symbol called the Laurent polynomial $a(z) = \sum_{i \in Z} a_i z^i$ of a mask $\{a_i\}, i \in Z$ with finite support. The corresponding symbols play an efficient role to analyze the convergence and smoothness of subdivision scheme.

With the symbol, Hassan *et al*. (2002) provided a sufficient and necessary condition for a uniform convergent subdivision scheme. A subdivision scheme S is uniform convergent if and only if there is an integer $L \geq 1$, such that $\left\| \left(\frac{1}{3} S_1\right)^L \right\|_{\infty} < 1$.

The subdivision scheme S_1 with symbol $a_1(z)$ is related to subdivision scheme S with symbol $a(z)$, where $a_1(z) = \frac{3z^2}{1+z+z^2}\, a(z)$. The subdivision scheme S with symbol $a(z)$ satisfies Eq. (1) then there exists a subdivision scheme S_1 with the property:

$$df^{k} = S_1 df^{k-1}, \quad k = 1, 2, \ldots$$

where, $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 3^k (f_{i+1}^{k} - f_i^k) : i \in Z\}$. The norm $\|S\|_{\infty}$ of a subdivision scheme S with a mask $\{a_i\}, i \in Z$ is defined by:

$$\|S\|_{\infty} = \max\{\sum_{i \in Z} |a_{3i}|, \sum_{i \in Z} |a_{3i+1}|, \sum_{i \in Z} |a_{3i+2}|\}$$

Theorem 1: Ternary 6-point approximating subdivision scheme defined in Eq. (1) converges and has smoothness C^5.

Proof: Consider the refinement equations defined in equation 1 and the Laurent polynomial a (z) for the mask of the proposed ternary 6-point approximating subdivision scheme can be written as:

$$\begin{aligned} a(z) = {} & \frac{1}{933120} z^{-9} + \frac{1}{3840} z^{-8} + \frac{625}{186624} z^{-7} + \frac{16801}{933120} z^{-6} + \frac{79}{1280} z^{-5} + \frac{142301}{933120} z^{-4} \\ & + \frac{135233}{466560} z^{-3} + \frac{841}{1920} z^{-2} + \frac{250213}{466560} z^{-1} + \frac{250213}{466560} z^{0} + \frac{841}{1920} z^{1} + \frac{135233}{466560} z^{2} \\ & + \frac{142301}{933120} z^{3} + \frac{79}{1280} z^{4} + \frac{16801}{933120} z^{5} + \frac{625}{186624} z^{6} + \frac{1}{3840} z^{7} + \frac{1}{933120} z^{8} \end{aligned}$$

Laurent polynomial method is used to prove the smoothness of the ternary 6-point approximating subdivision scheme to be C^5. Taking:

$$b^{[m,L]}(z) = \frac{1}{3^L} a_m^{[L]}(z), \quad L = 1, 2, \ldots, m$$

where,

$$a_m(z) = \left(\frac{3z^2}{1+z+z^2}\right) a_{m-1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)^m a(z)$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{3^j})$$

With a choice of $m=1$ and $L=1$, it can be written as:

$$b^{[1,1]}(z)=\frac{1}{3^1}a_1(z)=\frac{1}{933120}z^{-7}+\frac{121}{466560}z^{-6}+\frac{1441}{466560}z^{-5}+\frac{4559}{311040}z^{-4}+\frac{5129}{116640}z^{-3}+\frac{10949}{116640}z^{-2}+\frac{78921}{466560}z^{-1}+\frac{14941}{77760}z^{0}+\frac{14941}{77760}z^{1}+\frac{78921}{466560}z^{2}+\frac{10949}{116640}z^{3}+\frac{5129}{116640}z^{4}+\frac{4559}{311040}z^{5}+\frac{1441}{466560}z^{6}+\frac{121}{466560}z^{7}+\frac{1}{933120}z^{8}$$

To determine the convergence of subdivision scheme S, considering:

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[1,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S is convergent.

In order to prove the ternary 6-point approximating subdivision scheme developed to be C^1. Consider m = 2 and L = 1, the Laurent polynomial gives:

$$b^{[2,1]}(z)=\frac{1}{3^1}a_2(z)=\frac{1}{311040}z^{-5}+\frac{241}{311040}z^{-4}+\frac{11}{1296}z^{-3}+\frac{2699}{77760}z^{-2}+\frac{6899}{77760}z^{-1}+\frac{205}{1296}z^{0}+\frac{10841}{51840}z^{1}+\frac{10841}{51840}z^{2}+\frac{205}{1296}z^{3}+\frac{6899}{77760}z^{4}+\frac{2699}{77760}z^{5}+\frac{11}{1296}z^{6}+\frac{241}{311040}z^{7}+\frac{1}{311040}z^{8}$$

To determine the convergence of subdivision scheme S_1, considering:

$$\left\|\frac{1}{3}S_2\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[2,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S_1 is convergent and the subdivision scheme $S \in C^1$.

In order to prove the ternary 6-point approximating subdivision scheme developed to be C^2. Consider $m=3$ and $L=1$; the Laurent polynomial gives:

$$b^{[3,1]}(z)=\frac{1}{3^1}a_3(z)=\frac{1}{103680}z^{-3}+\frac{1}{432}z^{-2}+\frac{2399}{103680}z^{-1}+\frac{2719}{34560}z^{0}+\frac{71}{432}z^{1}+\frac{889}{3840}z^{2}+\frac{889}{3840}z^{3}+\frac{71}{432}z^{4}+\frac{2719}{34560}z^{5}+\frac{2399}{103680}z^{6}+\frac{1}{432}z^{7}+\frac{1}{103680}z^{8}$$

To determine the convergence of subdivision scheme S_2, considering:

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[3,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S_2 is convergent and the subdivision scheme $S \in C^2$.

In order to prove the ternary 6-point approximating subdivision scheme developed to be C^3. Consider $m=4$ and $L=1$; the Laurent polynomial gives:

$$b^{[4,1]}(z)=\frac{1}{3^1}a_4(z)=\frac{1}{34560}z^{-1}+\frac{239}{34560}z^{0}+\frac{2159}{34560}z^{1}+\frac{5759}{34560}z^{2}+\frac{4561}{17280}z^{3}+\frac{4561}{17280}z^{4}+\frac{5759}{34560}z^{5}+\frac{2159}{34560}z^{6}+\frac{239}{34560}z^{7}+\frac{1}{34560}z^{8}$$

To determine the convergence of subdivision scheme S_3, considering:

$$\left\|\frac{1}{3}S_4\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[4,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S_3 is convergent and the subdivision scheme $S \in C^3$.

In order to prove the ternary 6-point approximating subdivision scheme developed to be C^4. Consider $m=5$ and $L=1$; the Laurent polynomial gives:

$$b^{[5,1]}(z)=\frac{1}{3^1}a_5(z)=\frac{1}{11520}z^{1}+\frac{119}{5760}z^{2}+\frac{1}{6}z^{3}+\frac{3601}{11520}z^{4}+\frac{3601}{11520}z^{5}+\frac{1}{6}z^{6}+\frac{119}{5760}z^{7}+\frac{1}{11520}z^{8}$$

To determine the convergence of subdivision scheme S_4, considering:

$$\left\|\frac{1}{3}S_5\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[5,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S_4 is convergent and the subdivision scheme $S \in C^4$.

In order to prove the ternary 6-point approximating subdivision scheme developed to be C^5. Consider $m=6$ and $L=1$; the Laurent polynomial gives:

$$b^{[6,1]}(z)=\frac{1}{3^1}a_6(z)=\frac{1}{3840}z^{3}+\frac{79}{1280}z^{4}+\frac{841}{1920}z^{5}+\frac{841}{1920}z^{6}+\frac{79}{1280}z^{7}+\frac{1}{3840}z^{8}$$

To determine the convergence of subdivision scheme S_5, considering:

$$\left\|\frac{1}{3}S_6\right\|_\infty = \max\left\{\sum_\beta \left|b_{\gamma+3\beta}{}^{[6,1]}\right| : \gamma = 0,1,2\right\} = \max\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\} = \frac{1}{3} < 1$$

Therefore, the subdivision scheme S_5 is convergent and the subdivision scheme $S \in C^5$.

HOLDER REGULARITY OF THE SUBDIVISION SCHEME

Theorem 1: Implies that continuity of the proposed ternary 6-point approximating subdivision scheme is C^5. Based on generalized Rioul (1992), its highest degree of smoothness can be found. The subdivision scheme has Holder Regularity $R_H = 5 + v^k$ for all $k \geq 1$, where v^k is given by:

$$3^{-kv^k} = \left\| \left(\frac{1}{3} S_6\right)^k \right\|_\infty$$

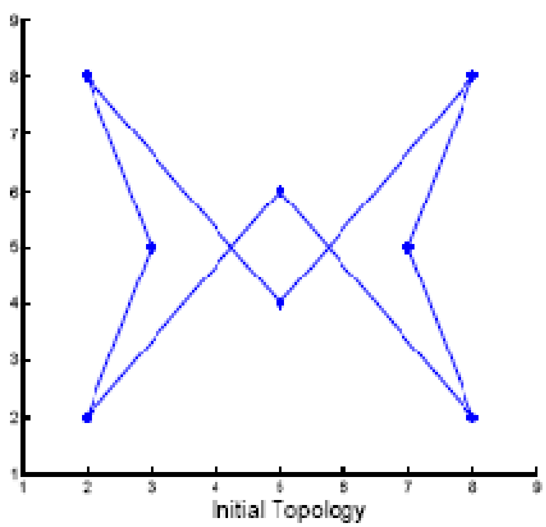
Following Rioul (1992), it is found that the Holder exponent for the ternary 6-point approximating subdivision scheme is $C^{5.75}$.

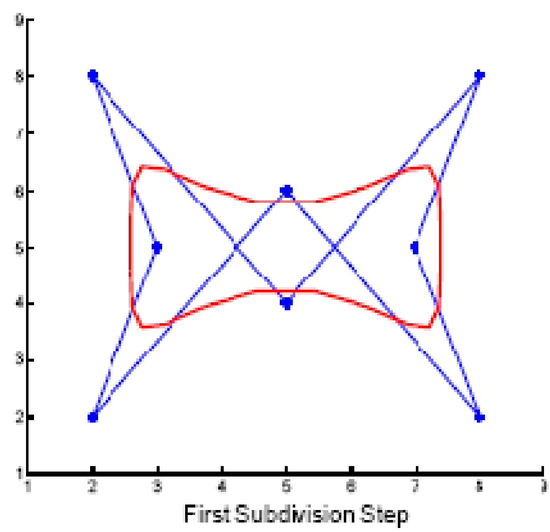
Comparison of subdivision schemes: For sketching the smooth curves and surfaces, the higher degree of smoothness of the subdivision scheme is required. The order of derivative continuity of the scheme depends on the mask used in the subdivision scheme. There are two ways to obtain a desired mask which gives maximum order of derivative continuity. One is to increase the support size of the subdivision scheme; the other is to find specific values of the mask which represents the bar centric combination of data points for a fixed support size of the scheme. Comparison Table 1 demonstrates that the higher degree of smoothness of the curve can be achieved with the increase of the support size of the ternary subdivision scheme. Ternary 2-point approximating subdivision scheme (2010) generates C^1 limiting curve with the support size 2.5. Ternary 3-point interpolating subdivision scheme (2002) generates C^1 limiting curve with the support size 4. Ternary 3-point approximating subdivision schemes (2002) and (2010) generates C^2 limiting curve with the support size 4. Ternary 4-point interpolating subdivision scheme (2002) generates C^2 limiting curve with the support size 5. Ternary 4-point approximating subdivision schemes (2007) and (2010) generates C^2 and C^3 limiting curve with the support size 5.5, respectively. Ternary 5-point approximating subdivision scheme (2009) generates C^4 limiting curve with the support size 7. A new ternary 6-point approximating subdivision scheme is introduced by increasing the support size of the ternary subdivision scheme that generates a limiting curve of C^5 continuity. Comparison Table 1. Illustrates the benefit of the proposed scheme that it generates the limiting curve of maximum degree of smoothness (i.e., C^5 continuity). Moreover, the proposed scheme observes the variation diminishing property.

Table 1: Comparison of the ternary subdivision schemes

Subdivision scheme	Type	Support (size)	Continuity
Ternary 2-point (2010)	Approximating	2.5	C^1
Ternary 3-point (2002)	Interpolating	4.0	C^1
Ternary 3-point (2002)	Approximating	4.0	C^2
Ternary 3-point (2010)	Approximating	4.0	C^2
Ternary 4-point (2002)	Interpolating	5.0	C^2
Ternary 4-point (2007)	Approximating	5.5	C^2
Ternary 4-point (2010)	Approximating	5.5	C^3
Ternary 5-point (2009)	Approximating	7.0	C^4
Proposed ternary 6-point	Approximating	8.5	C^5

Examples: The geometric behavior with two subdivision steps of the proposed ternary 6-point approximating subdivision scheme have been illustrated in four different examples as shown in Fig. 1 to 4. Four examples reveal the usefulness of the proposed scheme





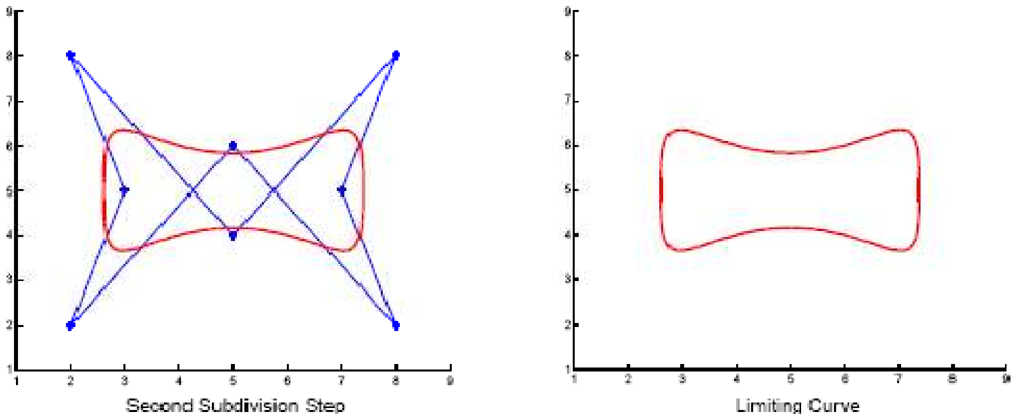


Fig. 1: A ternary 6-point approximating subdivision scheme with two subdivision steps and the limiting curve of C^5 continuity

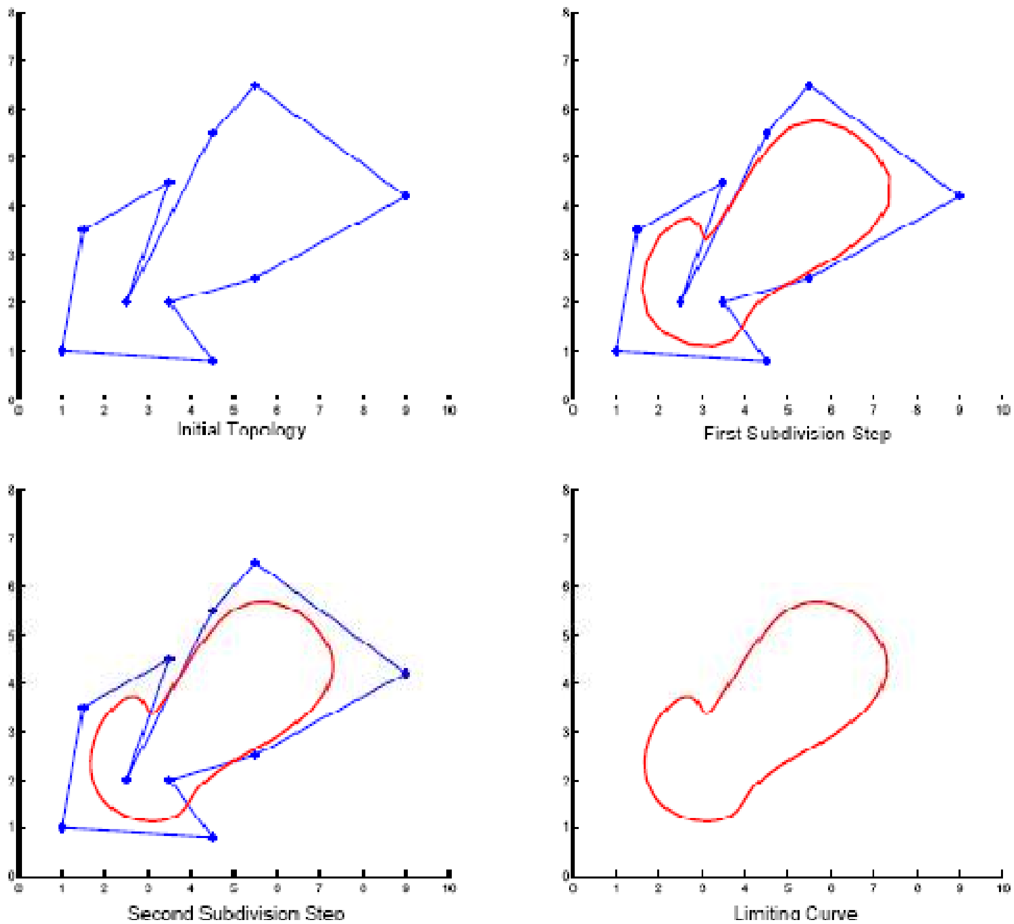


Fig. 2: A ternary 6-point approximating subdivision scheme with two subdivision steps and the limiting curve of C^5 continuity

that generates the smooth limiting curve of C^5 continuity and gives the pleasing results.

CONCLUSION

A univariate symmetric ternary 6-point approximating subdivision scheme has been introduced using quintic B-spline basis functions. The proposed scheme generates the limiting curve of C^5 continuity and its limiting functions has a support (-9, 8). The smoothness of the proposed subdivision scheme has been analyzed, using the Laurent polynomial method and Holder exponent of the subdivision scheme is $C^{5.75}$. The examples reveal that the limiting curve generated by the proposed scheme gives maximum degree of smoothness as compare to subsist subdivision schemes.

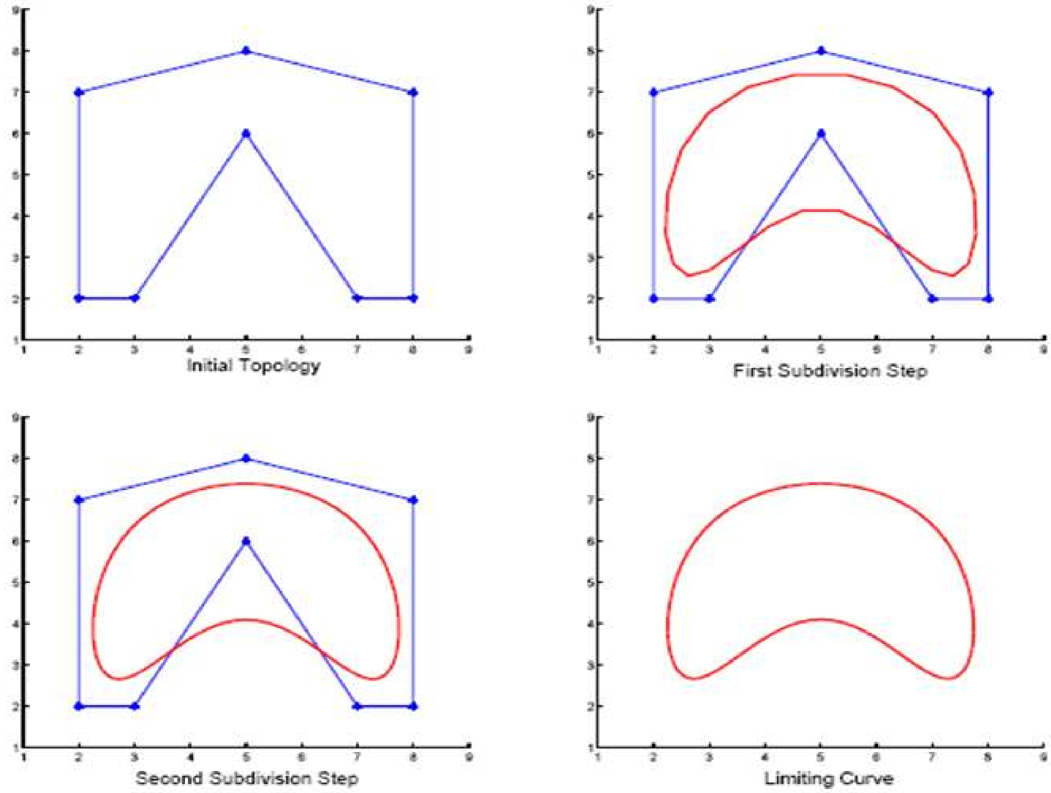


Fig. 3: A ternary 6-point approximating subdivision scheme with two subdivision steps and the limiting curve of C^5 continuity

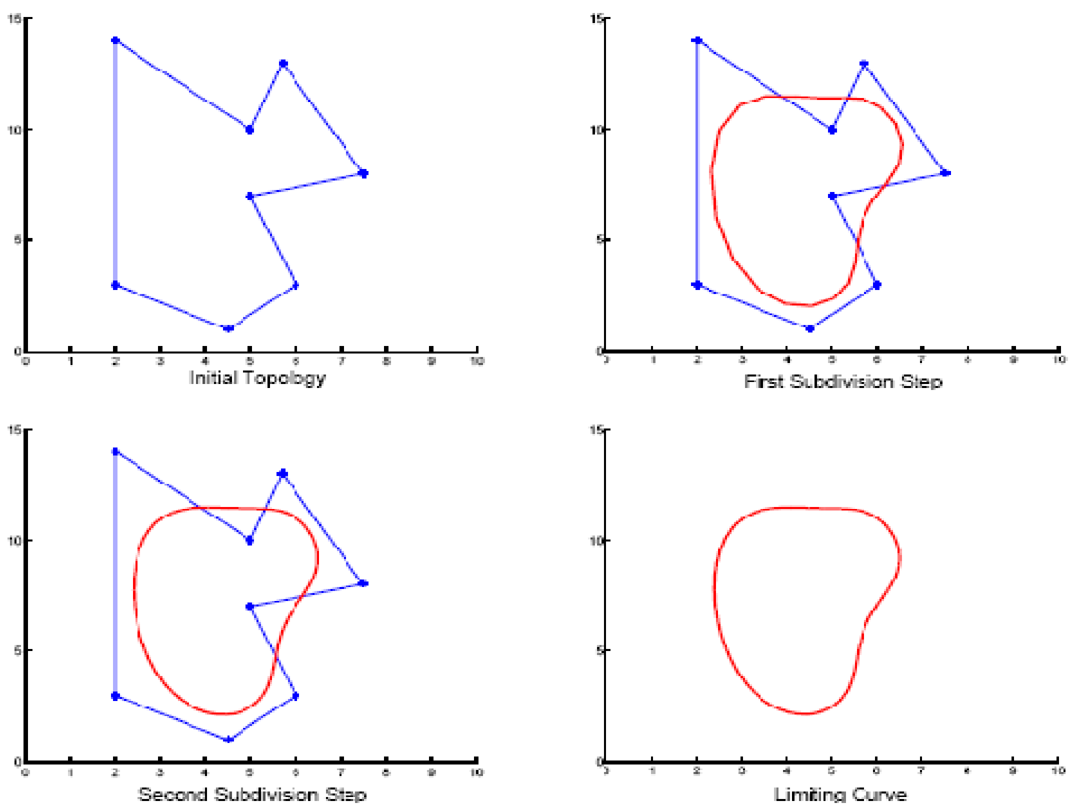


Fig. 4: A ternary 6-point approximating subdivision scheme with two subdivision steps and the limiting curve of C^5 continuity

REFERENCES

Chaikin, G.M., 1974. An algorithm for high speed curve generation. Comput. Vision Graph., 3(4): 346-349.

Hassan, M.F. and N.A. Dodgson, 2003. Ternary and Three Point Univariate Subdivision Schemes. Curve and Surface Fitting: Sant-Malo 2002 (Albert Cohen, Jean-Louis Merrien and Larry L. Schumaker, eds. Nashboro Press, Brentwood), pp: 199-208.

Hassan, M.F., I.P. Ivrissimitzis, N.A. Dodgson and M.A. Sabin, 2002. An interpolating 4-points C^2 ternary stationary subdivision scheme. Comput. Aided Geom. D., 19: 1-18.

Ko, K.P., B.G. Lee and G.J. Yoon, 2007. A ternary 4-point approximating subdivision scheme. Appl. Math. Comput., 190: 1563-1573.

Rioul, O., 1992. Simple regularity criteria for subdivision schemes. SIAM J. Math. Anal., 23: 1544-1576.

Siddiqi, S.S. and K. Rehan, 2009. A stationary ternary C^4 scheme for curve sketching. Eur. J. Sci. Res., 30(3): 380-388.

Siddiqi, S.S. and K. Rehan, 2010a. Improved binary four point subdivision scheme and new corner cutting scheme. Comput. Math. Appl., 59: 2647-2657.

Siddiqi, S.S. and K. Rehan, 2010b. Modified form of binary and ternary 3-point subdivision schemes. Appl. Math. Comput., 216: 970-982.

Siddiqi, S.S. and K. Rehan, 2010c. A ternary three point scheme for curve designing. Int. J. Comput. Math., 87(8): 1709-1715.

Zheng, H., Z. Ye, Z. Chen and H. Zhao, 2005. A controllable ternary interpolatory subdivision scheme. Int. J. CAD/CAM, 5(1).

Zheng, H., Z. Ye, Z. Chen and H. Zhao, 2007. Fractal range of a 3-point ternary interpolatory subdivision scheme with two parameters. Chaos, Soliton. Fract., 32: 1838-1845.