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Research Article

Some Characterizations of Intra-regular Abel-Grassmann Groupoids

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Abstract: In this study we introduce a new class of a non-associative algebraic structure namely intra-regular Abel Grassmann's groupoid (AG-groupoid in short). We apply generalized fuzzy ideal theory to this class and discuss its related properties. We introduce $(\in, \in \vee q_k)$-fuzzy semiprime ideals in AG-groupoids and characterize it. Specifically we have characterized intra-regular AG-groupoids in terms of left, bi and two sided ideals and $(\in, \in \vee q_k)$ -fuzzy left, bi and two sided ideals. For support of our arguments we give examples of AG-groupoids. At the end we characterize intra-regular AG-groupoids using the properties of $(\in, \in \vee q_k)$-fuzzy semiprime ideals.

Keywords: AG-groupoid, $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals, left invertive law, medial law, paramedial law

INTRODUCTION

The real world has a lot of different aspects which are not usually been specified. In different fields of knowledge like engineering, medical science, mathematics, physics, computer science and artificial intelligence, many problems are simplified by constructing their "models". These models are very complicated and it is impossible to find the exact solutions in many occasions. Therefore, the classical set theory, which is precise and exact, may not be suitable for such problems of uncertainty.

In today's world, many theories have been developed to deal with such uncertainties like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionist fuzzy sets and theory of rough sets. The theory of soft sets has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc. The basic concept of fuzzy set theory was first given by Zadeh (1965). Zadeh (1965) discussed the relationships between fuzzy set theory and probability theory. Rosenfeld (1971) initiated the fuzzy groups in fuzzy set theory. Mordeson *et al*. (2003) have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines.

The idea of belongingness of a fuzzy point to a fuzzy subset under the natural equivalence on a fuzzy subset has been defined by Murali (2004). Bhakat and Das (1992) gave the concept of (α, β) -fuzzy subgroups where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. The idea of $(\in, \in \vee q)$ -fuzzy subgroups is a generalization of fuzzy subgroupoid defined by Rosenfeld (1971). An $(\in, \in \vee q_k)$ -fuzzy bi-ideals and $(\in, \in \vee q_k)$- fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy ideals of a semi group are defined in Shabir *et al*. (2010a).

In this study we have discussed the $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals in a new non-associative algebraic structure, that is, in AG-groupoids and developed some new results. We have characterized intra-regular AG-groupoids by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

A groupoid is S called an AG-groupoid if it satisfies the left invertive law, that is:

$$(ab)c = (cb)a, \text{ for all } a, b, c \in S.$$

Every AG-groupoid satisfies the medial law:

$$(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S$$

It is basically a non-associative algebraic structure in between a groupoid and a commutative semi group. It is important to mention here that if an AG-groupoid contains identity or even right identity, then it becomes a commutative monoid. An AG-groupoid is not necessarily contains a left identity and if it contains a left identity then it is unique (Mushtaq and Yusuf, 1978). An AG-groupoid S with left identity satisfies the paramedial law, that is:

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$$(ab)(cd) = (db)(ca), \text{for all } a,b,c,d \in S$$

Also S satisfies the following law:

$$a(bc) = b(ac), \text{for all } a,b,c,d \in S.$$

If S is an AG-groupoid with left identity e then $Sa = \{sa : s \in S\}$ is both left and bi-ideal of S containing a. Moreover $S = eS \subseteq S^2$. Therefore, $S = S^2$.

For a subset A of an AG-groupoid S the characteristics function, C_A is defined by:

$$C_A = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}.$$

It is important to note that an AG-groupoid can also be considered as a fuzzy subset of itself and we can write $S = C_S$, i.e., $S(x) = 1$, for all x in S.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by:

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} \{f(b) \wedge g(c)\}, \text{if there exist } b,c \in S, \text{such that } a = bc. \\ 0, \text{ otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of S:

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } S.$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x), \text{ for all } x \text{ in } S$$

The following definitions for AG-groupoids are same as for semigroups in Shabir *et al*. (2010b):

Definition 1: A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy AG-sub groupoid of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies, $x_t \in f$, $y_r \in f$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k f$

Definition 2: A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies, $x_t \in f$ implies $(yx)_t \in \vee q_k f$ $\left(x_t \in f \text{ implies } (xy)_t \in \vee q_k f\right)$.

Definition 3: A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$, for all $x, y, z \in S$ and $t, r \in (0,1]$.

Definition 4: A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following conditions hold:

- If $x_t \in f$ and $y_r \in S$ implies $(xy)_{\min\{t,r\}} \in \vee q_k f$
- If $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$

Definition 5: A fuzzy subset f of an AG-groupoid S is said to be $(\in, \in \vee q)$-fuzzy semiprime if it satisfies $x_t^2 \in f \Rightarrow x_t \in \vee q f$ for all $x \in S$ and $t \in (0,1]$.

Definition 6: Let A be any subset of an AG-groupoid S, then the characteristic function $(C_A)_k$ is defined as:

$$(C_A)_k(a) = \begin{cases} \geq \frac{1-k}{2} \text{ if } a \in A \\ 0 \quad \text{otherwise.} \end{cases}$$

The proofs of the following four theorems are same as in Shabir *et al*. (2010b).

Theorem 1: Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$.

Theorem 2: A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if:

$$f(xy) \geq \min\{f(y), \tfrac{1-k}{2}\}\left(f(xy) \geq \min\{f(x), \tfrac{1-k}{2}\}\right)$$

Theorem 3: Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if:

- $(i)\, f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ for all $x, y \in S$
- $(ii)\, f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$ for all $x, y, z \in S$

Theorem 4: A fuzzy subset f of an AG-groupoid S is $(\in, \in \vee q_k)$-fuzzy semiprime if and only if $f(x) \geq f(x^2) \wedge \frac{1-k}{2}$ for all $x \in S$.

Proof: Let f be a fuzzy subset of an AG-groupoid S which is $(\in, \in \vee q_k)$-fuzzy semiprime. If there exists some $x_0 \in S$ such that $f(x_0) < t_0 = f(x_0^2) \wedge \frac{1-k}{2}$. Then $(x_0^2)_{t_0} \in f$, but $(x_0)_{t_0} \overline{\in} f$. In addition, we have $(x_0)_{t_0} \in \vee q_k f$ since f is $(\in, \in \vee q_k)$-fuzzy semiprime. On the other hand, we have $f(x_0) + t_0 \leq t_0 + t_0 \leq 1$. Thus

$(x_0)_{t_0}\overline{q_k}f$ and so $(x_0)_{t_0}\overline{\in\vee q_k}f$. This is a contradiction. Hence $f(x)\geq f(x^2)\wedge\frac{1-k}{2}$ for all $x\in S$.

Conversely, assume that f is a fuzzy subset of an AG-groupoid S such that $f(x)\geq f(x^2)\wedge\frac{1-k}{2}$ for all $x\in S$. Let $x_t^2\in f$. Then $f(x^2)\geq t$ and so $f(x)\geq f(x^2)\wedge\frac{1-k}{2}\geq t\wedge\frac{1-k}{2}$. Now, we consider the following two cases:

- If $t\leq\frac{1-k}{2}$, then $f(x)\geq t$. That is, $x_t\in f$. Thus we have $x_t\in\vee q_k f$.
- If $t>\frac{1-k}{2}$, then $f(x)\geq\frac{1-k}{2}$. It follows that $f(x)+t\geq\frac{1-k}{2}+t>1$. That is, $x_t q_k f$ and so $x_t\in\vee q_k f$ also holds. Therefore, we conclude that f is $(\in,\in\vee q_k)$ fuzzy semiprime as required.

Example 1: Let $S=\{1,2,3\}$, then from the following multiplication table one can easily verify that S is an AG- groupoid:

.	1	2	3
1	1	1	1
2	3	3	1
3	1	1	1

Let us define fuzzy subset f of S as: $f(1)=0.9, f(2)=0.6, \quad f(3)=0.8$. Then f is clearly an $(\in,\in\vee q_k)$ -fuzzy ideal.

Definition 7: An AG groupoid S is called intra-regular AG-groupoid if for each a in S there exists x, y in S such that $a=(xa^2)y$.

Example 2: Let $S=\{1,2,3,4,5,6\}$, the following table shows that S is an intra-regular AG-groupoid:

$*$	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	4	5	6	3
4	1	2	3	4	5	6
5	1	2	6	3	4	5
6	1	2	5	6	3	4

It is easy to see that $(S,*)$ is an AG-groupoid and is non-commutative and non-associative structure because $(3*4)\neq(4*3)$ and $(3*6)*4\neq 3*(6*4)$. Also:

$$1=(3*1^2)*1,\ 2=(2*2^2)*2,\ 3=(4*3^2)*6,$$
$$4=(4*4^2)*4,\ 5=(6*5^2)*3,\ 6=(5*6^2)*5$$

Therefore, $(S,*)$ is an intra-regular AG-groupoid. Clearly {1} and {1, 2} are ideals of S. A fuzzy subset $f: S\rightarrow[0,1]$ is defined as:

$$f(x)=\begin{cases}0.9 \text{ for } x=1\\ 0.8 \text{ for } x=2\\ 0.7 \text{ for } x=3\\ 0.6 \text{ for } x=4\\ 0.5 \text{ for } x=5\\ 0.5 \text{ for } x=6\end{cases}$$

Then clearly f is an $(\in,\in\vee q_k)$ -fuzzy bi-ideal of S. Also f is $(\in,\in\vee q_k)$ -fuzzy semiprime.

Lemma 1: Let A be a non-empty subset of an AG-groupoid S, then:

- A is a left (right, two-sided) ideal of S if and only if $(C_A)_k$ is an $(\in,\in\vee q_k)$ fuzzy left (right, two-sided) ideal of S.
- A of an AG-groupoid S with left identity is bi-ideal if and only if $(C_A)_k$ is $(\in,\in\vee q_k)$-fuzzy bi-ideal.

Proof: It is same as in Shabir *et al*. (2010a).

Lemma 2: Let A and B be non-empty subsets of an AG-groupoid S, then the following properties hold:

- $(C_{A\cap B})_k=(C_A\wedge_k C_B)$.
- $(C_{A\cup B})_k=(C_A\vee_k C_B)$.
- $(C_{AB})_k=(C_A\circ_k C_B)$.

Proof: It is same as in Shabir *et al*. (2010a).

Theorem 5: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular
(ii) For every left ideal L and for every ideal I, $L\cap I=IL$.
(iii) For every $(\in,\in\vee q_k)$-fuzzy left ideal f and $(\in,\in\vee q_k)$ -fuzzy ideal g $f\wedge_k g=g\circ_k f$.

Proof: $(i)\Rightarrow(iii)$ Assume that S is an intra-regular AG-groupoid and f and g are $(\in,\in\vee q_k)$ -fuzzy left and

$(\in, \in \vee q_k)$-fuzzy ideal of *S*. Since *S* is intra-regular therefore for any a in *S* there exist x, y in S such that $a = (xa^2)y$. By using (4) and (1):

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

So for any *a* in *S* there exist *u* and *v* in *S* such that $a = uv$, then:

$$\begin{aligned}(g \circ_k f)(a) &= \bigvee_{a=uv} g(u) \wedge f(v) \wedge \frac{1-k}{2} \\ &\geq g(y(xa)) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq g(xa) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= (g \wedge f)(a) \wedge \frac{1-k}{2} \\ &= (f \wedge g)(a) \wedge \frac{1-k}{2} \\ &= (f \wedge_k g)(a).\end{aligned}$$

This implies that:

$$f \wedge_k g \leq g \circ_k f.$$

Now,

$$\begin{aligned}(g \circ_k f)(a) &= \bigvee_{a=bc} g(b) \wedge f(c) \wedge \frac{1-k}{2} \\ &\leq \bigvee_{a=bc} g(bc) \wedge f(bc) \wedge \frac{1-k}{2} \\ &= g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= (g \wedge f)(a) \wedge \frac{1-k}{2} \\ &= (f \wedge_k g)(a).\end{aligned}$$

This implies that:

$$g \circ_k f \leq f \wedge_k g.$$

Therefore,

$$g \circ_k f = f \wedge_k g.$$

$(iii) \Rightarrow (ii)$ Let L be the left and I be an ideal of *S*. Then $(C_L)_k$ and $(C_I)_k$ are the $(\in, \in \vee q_k)$ fuzzy left and $(\in, \in \vee q_k)$ fuzzy ideal of S respectively. Therefore $(C_{L\cap I})_k = (C_L \wedge_k C_I) \leq C_I \circ_k C_L = (C_{IL})_k$.

This implies $(C_{L\cap I})_k \leq (C_{IL})_k$. Thus $L \cap I \subseteq IL$.

$(ii) \Rightarrow (i)$ By using (2),(3) and (4), we get:

$$\begin{aligned}(Sa^2)S &= (Sa^2)(SS) = (SS)(a^2S) = S(a^2S) = a^2(SS) \\ &= (aa)(SS) = (Sa)(Sa) = (SS)(aa) \subseteq Sa^2.\end{aligned}$$

This implies $(Sa^2)S \subseteq Sa^2$. Also:

$$S(Sa^2) = (SS)(Sa^2) = (a^2S)(SS) \subseteq (a^2S)S = (SS)a^2 = Sa^2.$$

This implies $S(Sa^2) \subseteq Sa^2$. since Sa^2 is both left and right ideal therefore it is an ideal containing a^2. Since Sa^2 is semiprime therefore $a \in Sa^2$. Now by using (ii):

$$a \in Sa \cap Sa^2 = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

Hence *S* is intra-regular.

Lemma 3: Every $(\in, \in \vee q_k)$-fuzzy left ideal of an AG-groupoid *S* is $(\in, \in \vee q_k)$-fuzzy bi-ideal.

Proof: Let *S* be an AG-groupoid and *f* be an $(\in, \in \vee q_k)$-fuzzy left ideal of *S*. Then for any x in *S* there exist *a* and *b* in *S* such that:

$$\begin{aligned}f((ax)b) &\geq f(b) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge f(b) \wedge \frac{1-k}{2}.\end{aligned}$$

Hence *f* is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of *S*

Theorem 6: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) *S* is intra-regular
(ii) For every bi-ideal *B* and left ideal L $B \cap L \subseteq BL$.
(iii) For every $(\in, \in \vee q_k)$-fuzzy bi-ideal *F* and $(\in, \in \vee q_k)$-fuzzy left ideal g, $f \wedge_k g \leq f \circ_k g$.
(iv) For every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal *f* and every $(\in, \in \vee q_k)$-fuzzy left ideal *g*, $f \wedge_k g \leq f \circ_k g$.
(v) For all $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal *f* and *g*, $f \wedge_k g \leq f \circ_k g$.

Proof: $(i) \Rightarrow (v)$ Assume that S is an intra-regular AG-groupoid with left identity and *f* and *g* are $(\in, \in \vee q_k)$-generalized fuzzy bi-ideal of *S* respectively. Thus, for any a in S there exist u and v in S such that $a=uv$, then:

$$(f\circ_k g)(a)=\bigvee_{a=uv} f(u)\wedge g(v)\wedge\frac{1-k}{2}.$$

Since S is intra-regular so for any a in S there exist $x,y\in S$ such that $a=(xa^2)y.$ by using (4) and (1),(3) and (2) we get:

$$\begin{aligned}a&=(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a\\&=(y(x((xa^2)y)))a=(y((xa^2)(xy)))a=((xa^2)(y(xy)))a\\&=((x(aa))(y(xy)))a=((a(xa))(y(xy)))a\\&=(((y(xy))(xa))a)a=(((y(xy))(x((xa^2)y)))a)a\\&=(((y(xy))((xa^2)(xy)))a)a=(((xa^2)((y(xy))(xy)))a)a\\&=((((xy)a^2)((y(xy))x))a)a=((((xy)(y(xy)))(a^2x))a)a\\&=((a^2(((xy)(y(xy)))x))a)a=(((aa)(((xy)(y(xy)))x))a)a\\&=(((x((xy)(y(xy))))(aa))a)a=((a((x(xy)(y(xy)))a))a)a.\end{aligned}$$

Thus, we have:

$$\begin{aligned}(f\circ_k g)(a)&=\bigvee_{a=uv} f(u)\wedge g(v)\wedge\frac{1-k}{2}\\&\geq f((a(((x(xy)(y(xy)))a))a)\wedge g(a)\wedge\frac{1-k}{2}\\&\geq (f(a)\wedge f(a))\wedge g(a)\wedge\frac{1-k}{2}\\&=(f\wedge g)(a)\wedge\frac{1-k}{2}.\\&=(f\wedge_k g)(a).\end{aligned}$$

This implies that $f\wedge_k g\leq f\circ_k g$.

$(v)\Rightarrow(iv)$ Let f be an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal and g be an $(\in,\in\vee q_k)$-fuzzy left ideal. Then by Lemma left ideal is a bi-ideal, g is also an $(\in,\in\vee q_k)$-fuzzy bi-ideal therefore (iv) is obvious.

$(iv)\Rightarrow(iii)$ is obvious.

$(iii)\Rightarrow(ii)$ Let B be a bi-ideal and L be a left ideal of S. Then $(C_B)_k$ and $(C_L)_k$ are $(\in,\in\vee q_k)$-fuzzy bi-ideal and $(\in,\in\vee q_k)$ -fuzzy left ideal of S. Then we get $(C_{B\cap L})_k=C_B\wedge_k C_L\leq C_B\circ_k C_L=(C_{BL})_k$.

Thus $B\cap L\subseteq BL$.

$(ii)\Rightarrow(i)$ Since Sa is both bi-ideal and left ideal containing a. Therefore by (ii) and using (3),(2) and (4), we obtain:

$$\begin{aligned}a&\in Sa\cap Sa\subseteq(Sa)(Sa)=(aa)(SS)=(aS)(aS)=a^2(SS)\\&=(aa)(SS)=S(a^2S)=(SS)(a^2S)=(Sa^2)(SS)=(Sa^2)S.\end{aligned}$$

Hence S is an intra-regular AG-groupoid.

Theorem 7: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular**.**
(ii) For all left ideals $A,B,\quad A\cap B\subseteq BA.$
(iii) For all $(\in,\in\vee q_k)$-fuzzy left ideals f and g, $f\wedge_k g\leq g\circ_k f.$
(iv) For all $(\in,\in\vee q_k)$-fuzzy bi-ideals f and g, $f\wedge_k g\leq g\circ_k f.$
(v) For all $(\in,\in\vee q_k)$- generalized fuzzy bi-ideals f and g, $f\wedge_k g\leq g\circ_k f.$

Proof: $(i)\Rightarrow(v)$ Let S be an intra-regular AG-groupoid and f and g are both $(\in,\in\vee q_k)$ -fuzzy bi-ideals. For any $a\in S$, there exist u and v in S such that $a=uv$, then we get:

$$\begin{aligned}(g\circ_k f)(a)&=\bigvee_{a=uv} g(u)\wedge f(v)\wedge\frac{1-k}{2}\\&\geq g((a(((x(xy)(y(xy)))a))a)\wedge f(a)\wedge\frac{1-k}{2}\\&\geq (g(a)\wedge g(a))\wedge f(a)\wedge\frac{1-k}{2}\\&\geq g(a)\wedge f(a)\wedge\frac{1-k}{2}\\&=(f\wedge_k g)(a).\end{aligned}$$

This implies $f\wedge_k g\leq g\circ_k f$.

$(v)\Rightarrow(iv)$ is obvious.

$(iv)\Rightarrow(iii)$ Let f and g are $(\in,\in\vee q_k)$-fuzzy left ideals. Then by Lemma left ideal is a bi-ideal, f and g are $(\in,\in\vee q_k)$ -fuzzy bi-ideals. Then (iii) is obvious.

$(iii)\Rightarrow(ii)$ Assume that A and B are the left ideals of S then $(C_A)_k$ and $(C_B)_k$ are $(\in,\in\vee q_k)$-fuzzy bi-ideals of S. Then we get:

$$(C_{A\cap B})_k=(C_A\wedge_k C_B)\leq C_B\circ_k C_A=(C_{BA})_k.$$

Thus $A\cap B\subseteq BA.$

$(ii)\Rightarrow(i)$ Since Sa is both bi-ideal and left ideal containing a. Using (ii), we get:

$$a\in Sa\cap Sa\subseteq(Sa)(Sa)=Sa^2=(Sa^2)S.$$

Hence S is intra-regular.

Theorem 8: Let S is an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular.
(ii) For every $(\in,\in\vee q_k)$-fuzzy left ideals f, g and $(\in,\in\vee q_k)$-fuzzy right ideal h, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h.$
(iii) For every $(\in,\in\vee q_k)$-fuzzy left ideals f, g and $(\in,\in\vee q_k)$-fuzzy bi ideal h, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h.$
(iv) For all $(\in,\in\vee q_k)$-fuzzy bi-ideals f, g and h, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h.$
(v) For all $(\in,\in\vee q_k)$-generalized fuzzy bi-ideals f, g and h, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h.$

Proof: $(i)\Rightarrow(iii)$ Assume that S is an intra-regular AG-groupoid and f and g are $(\in,\in\vee q_k)$-left ideals and h is an $(\in,\in\vee q_k)$-bi-ideal of S. Since S is an intra-regular AG-groupoid therefore for all $a\in S$ there exist x,y in S such that $a=(xa^2)y.$ By using (4) and (1), we get:

$$\begin{aligned}a&=(x(aa))y=(a(xa))y=(y(xa))a.\\&=(y(x((xa^2)y)))a=(y((xa^2)(xy)))a\\&=((xa^2)(y(xy)))a=((x(aa))(y(xy)))a\\&=((a(xa))(y(xy)))a=(((y(xy))(xa))a)a\\&=(((ax)((xy)y))a)a=(((((xy)y)x)a)a)a\\&=(((((xy)y)x)((xa^2)y)))a)a=(((xa^2)(((xy)y)x)y))a)a\\&=(((a(xa))(((xy)y)x)y))a)a=((((((xy)y)x)y)(xa))a)a)a\\&=((((y(((xy)y)x))(ax))a)a)a=(((a(y(((xy)y)x))x)a)a)a.\end{aligned}$$

Now we have:

$$\begin{aligned}((f\circ_k g)\circ_k h)(a)&=\bigvee_{a=uv}(f\circ_k g)(u)\wedge h(v)\wedge\frac{1-k}{2}\\&=\bigvee_{a=uv}\left(\left\{\bigvee_{u=pq}f(p)\wedge g(q)\wedge\frac{1-k}{2}\right\}\wedge h(v)\wedge\frac{1-k}{2}\right)\\&=\bigvee_{a=(pq)v}\left(f(p)\wedge g(q)\wedge h(v)\wedge\frac{1-k}{2}\right)\\&=\bigvee_{a=(((a(y(((xy)y)x))x)a)a)a=(pq)v}\left(f(p)\wedge g(q)\wedge h(v)\wedge\frac{1-k}{2}\right)\\&\geq(f(a)\wedge f(a))\wedge g(a))\wedge h(a)\wedge\frac{1-k}{2}\\&\geq((f(a)\wedge\frac{1-k}{2})\wedge g(a))\wedge h(a)\wedge\frac{1-k}{2}\\&=((f(a)\wedge g(a)\wedge\frac{1-k}{2})\wedge h(a))\wedge\frac{1-k}{2}\\&=(f\wedge_k g)\wedge h(a)\wedge\frac{1-k}{2}\\&=((f\wedge_k g)\wedge_k h)(a).\end{aligned}$$

This implies $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h.$
$(v)\Rightarrow(iv)$ is obvious.
$(iv)\Rightarrow(iii)$ Assume that f and g are $(\in,\in\vee q_k)$-fuzzy left ideals and h is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S then f and g are $(\in,\in\vee q_k)$-fuzzy bi-ideal, therefore (iii) is obvious.
$(iii)\Rightarrow(ii)$ is obvious.
$(ii)\Rightarrow(i)$ Assume that f,g are left ideals and S is a right ideal. Then by using (ii), we get:

$$f\wedge_k g=f\wedge_k g\wedge_k S\leq(g\circ_k f)\circ_k S\leq g\circ_k f.$$

Therefore, $f\wedge_k g\leq g\circ_k f$. Hence S is intra-regular.

Theorem 9: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular.
(ii) Every ideal of S is semiprime.
(iii) Every bi-ideal of S is semiprime.
(iv) Every $(\in,\in\vee q_k)$-fuzzy bi-ideal of S is fuzzy semiprime.
(v) Every $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal of S is fuzzy semiprime.

Proof: $(i)\Rightarrow(vii)$ Let S be an intra-regular and f be an $(\in,\in\vee q_k)$-generalized bi-ideal of S. Then for all $a\in S$ there exists x,y in S such that $a=(xa^2)y$. By using $(4),(1),(2)$ and (3), we get:

$$\begin{aligned}a&=(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a\\&=(y(x((xa^2)y))a=(x(y((xa^2)y)))a\\&=(x((xa^2)y^2))a=((xa^2)(xy^2))a\\&=(x^2(a^2y^2))a=(a^2(x^2y^2))a=(a(x^2y^2))a^2\\&=(((xa^2)y)(x^2y^2))a^2=((y^2.y)(x^2(xa^2)))a^2\\&=((y^2x^2)(y(xa^2)))a^2=((y^2x^2)((uv)(xa^2)))a^2\\&=((y^2x^2)((a^2v)(xu)))a^2=((y^2x^2)((a^2x)(vu)))a^2\\&=((((y^2x^2)(vu))x)(aa)))a^2=((ax)(a((y^2x^2)(vu))))a^2\\&=((aa)(x((x^2y^2)(vu))))a^2=(a^2((x((x^2y^2)(vu))))a^2.\end{aligned}$$

Thus we have:

$$f(a)=f((a^2((x((x^2y^2)(vu))))a^2)\geq f(a^2)\wedge f(a^2)=f(a^2).$$

Therefore, $f(a)\geq f(a^2).$
$(v)\Rightarrow(iv)$ is obvious.
$(iv)\Rightarrow(iii)$ Let B be a bi-ideal of S, then $(C_B)_k$ is an $(\in,\in\vee q_k)$ fuzzy bi-ideal of S. Let $a^2\in B$ then since $(C_B)_k$ is an $(\in,\in\vee q_k)$ fuzzy bi-ideal, therefore by

(iv), $(C_B(a))_k \geq (C_B(a^2))_k$, as $a^2 \in B$ so, $(C_B(a^2))_k = 1 \leq (C_B(a))_k$ this implies $(C_B(a))_k = 1$. Thus $a \in B$. Hence, B is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal is semiprime and since Sa^2 is an ideal containing a^2. Thus $a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S$. Hence S is an intra-regular AG-groupoid.

Theorem 10: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular.
(ii) For every left ideal L and bi-ideal B, $L \cap B \subseteq (LB)L$.
(iii) For every $(\in, \in \vee q_k)$-fuzzy left ideal f and $(\in, \in \vee q_k)$-fuzzy bi-ideal g $f \wedge_k g \leq (f \circ_k g) \circ_k f$.

Proof: $(i) \Rightarrow (iii)$ Let S be an intra-regular AG-groupoid and f be an $(\in, \in \vee q_k)$-fuzzy left ideal and g be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Since S is an intra-regular AG-groupoid then for any $a \in S$, there exists $x, y \in S$ such that $a = (xa^2)y$. Then by using (4) and (1), we get:

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(x((xa^2)y))a = (x(y((a(xa)y)))a \\ &= (x(a(xa))y^2)a = ((a(xa))(xy^2))a = (((xy^2)(xa))a)a. \end{aligned}$$

Therefore,

$$\begin{aligned} ((f \circ_k g) \circ_k f)(a) &= \bigvee_{a=uv} (f \circ_k g)(u) \wedge f(v) \wedge \frac{1-k}{2} \\ &\geq (f \circ_k g)(((xy^2)(xa))a) \wedge f(a). \end{aligned}$$

Since f is an $(\in, \in \vee q_k)$-fuzzy left ideal, therefore,

$$\begin{aligned} (f \circ_k g)(((xy^2)(xa))a) &= \bigvee_{((xy^2)(xa))a=rs} f(r) \wedge g(s) \wedge \frac{1-k}{2} \\ &\geq f((xy^2)(xa)) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(xa) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Thus,

$$((f \circ_k g) \circ_k f)(a) \geq f(a) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge g(a) \wedge \frac{1-k}{2}.$$

Hence $f \wedge_k g \leq ((f \circ_k g) \circ_k f)$.

$(iii) \Rightarrow (ii)$ Let L be a left ideal and B be a bi-ideal of S. Then (iii), $(C_L)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy left and $(\in, \in \vee q_k)$ -fuzzy bi-ideal of S. Then,

$$(C_{L \cap B})_k = C_L \wedge_k C_B \leq (C_L \circ_k C_B) \circ_k C_L \leq ((C_{LB})_k \circ_k C_L) = (C_{(LB)L})_k.$$

Therefore, we get $L \cap B \subseteq (LB)L$.

$(ii) \Rightarrow (i)$ Since Sa is both left and bi-ideal. Let $a \in S$. So By using (ii):

$$a \in Sa \cap Sa = ((Sa)(Sa))Sa = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

Therefore, S is an intra-regular AG-groupoid.

The proofs of following two lemmas are easy and therefore omitted.

Lemma 4: For any fuzzy subset f of an AG-groupoid S, $S \circ_k f \leq f$ and for any fuzzy right ideal g, $g \circ_k S \leq g$.

Lemma 5: Let S be an intra-regular AG-groupoid then for any $(\in, \in \vee q_k)$-fuzzy subsets f and g, $f \wedge_k g \wedge_k S = f \wedge_k g$.

Lemma 6: Let S be an intra-regular AG-groupoid then for any $(\in, \in \vee q_k)$-*fuzzy*-subsets f and g, $(g \circ_k f) \circ_k S \leq g \circ_k f$.

Proof: Let S be an intra-regular AG-groupoid and f and g are any $(\in, \in \vee q_k)$ -fuzzy-subsets then we get:

$$(g \circ_k f) \circ_k S = (S \circ_k f) \circ_k g \leq f \circ_k g = g \circ_k f.$$

Hence $(g \circ_k f) \circ_k S \leq g \circ_k f$.

Theorem 11: Let S be an intra-regular AG-groupoid and f and g are $(\in, \in \vee q_k)$-fuzzy ideals of S, then $f \circ_k g = f \wedge_k g$.

Proof: Let S be an intra-regular AG-groupoid and f and g are $(\in, \in \vee q_k)$-fuzzy ideals of S. Then for any a in S there exist x and y in S such $a = (xa^2)y$, then:

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

Now,

$$(f \circ_k g)(a) = \bigvee_{a=uv} \left\{ f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\}$$
$$\geq f(y(xa)) \wedge g(a) \wedge \frac{1-k}{2}$$
$$\geq .f(xa) \wedge g(a) \wedge \frac{1-k}{2}$$
$$\geq f(a) \wedge g(a) \wedge \frac{1-k}{2}$$
$$= (f \wedge g)(a) \wedge \frac{1-k}{2}$$
$$= (f \wedge_k g)(a).$$

This implies that $f \circ_k g \geq f \wedge_k g.$
Now,

$$(f \circ_k g)(a) = \bigvee_{a=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{a=uv} f(uv) \wedge g(uv) \wedge \frac{1-k}{2}$$
$$= f(a) \wedge g(a) \wedge \frac{1-k}{2}$$
$$= (f \wedge g)(a) \wedge \frac{1-k}{2}$$
$$= (f \wedge_k g)(a).$$

This implies that $f \circ_k g \leq f \wedge_k g.$ Hence $f \circ_k g = f \wedge_k g.$

Theorem 12: Let S be an AG-groupoid with left identity then the following conditions are equivalent:

(i) S is intra-regular.
(ii) For every ideals A and B, $AB \subseteq BA,$ and A and B are semiprime
(iii) For every $(\in, \in \vee q_k)$ -fuzzy ideals f and g, $f \circ_k g \leq g \circ_k f,$ and f and g are semiprime ideals.

Proof: $(i) \Rightarrow (iii)$ Let S be an intra-regular AG-groupoid and f and g are $(\in, \in \vee q_k)$-fuzzy ideals of S. Then we get:

$$f \circ_k g = f \wedge_k g = (f \wedge_k g) \wedge_k S = (g \circ_k f) \wedge_k S = S \circ_k (g \circ_k f) \leq g \circ_k f.$$

This implies $f \circ_k g \leq g \circ_k f.$ Also we will show that f and g are semiprime ideals. So, $f(a) = f((xa^2)y) \geq f(a^2).$
Thus $f(a) \geq f(a^2)$. Similarly $g(a) \geq g(a^2)$.

$(iii) \Rightarrow (ii)$ Let A and B are ideals then by Lemma A is ideal if C (A) k is fuzzy ideal, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$ -fuzzy ideals, therefore $(C_{AB})_k = C_A \circ_k C_B \leq C_B \circ_k C_A = (C_{BA})_k.$
Therefore we get $AB \subseteq BA$.
$(ii) \Rightarrow (i)$ Let $a^2 \in Sa^2.$ Since Sa^2 is semiprime therefore, $a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S.$
Hence, S is an intra-regular AG-groupoid.

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