Research Journal of Applied Sciences, Engineering and Technology 7(16): 3342-3347, 2014
DOI:10.19026/rjaset.7.679
ISSN: 2040-7459; e-ISSN: 2040-7467

Submitted: September 28, 2013 Accepted: November 19, 2013 Published: April 25, 2014

Research Article

Oscillation Criteria for Ordinary Differential Equations of Second Order Nonlinear with Alternating Coefficients

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Abstract: Several new oscillation criteria are established under quite general assumptions. Our results generalize and extend some earlier results of the literature. Examples are given to illustrate the results. We employ the averaging technique to obtain sufficient conditions for oscillation in the solutions of our equation.

Keywords: Averaging technique, nonlinear differential equations, oscillation criteria, Riccati substitution, second order

INTRODUCTION

The study of the oscillation of second order nonlinear ordinary differential equations with alternating coefficients is of special interest because of the fact that many physical systems are modeled by second-order nonlinear ordinary differential equations. For example, the so called Emden-Fowler equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of relativistic mechanics, nuclear physics and in the study of chemically reacting systems. The averaging techniques are used also in study of the nonlinear oscillations.

This study is concerned with the oscillation of solutions of the second order nonlinear differential equation:

$$(r(t)\psi(x(t))f(x'(t)))'+q(t)g(x(t))=H(t,x'(t),x(t)), \quad (1)$$

where,

- q and r : Continuous functions on the interval (t_0, ∞), $t_0 \geq 0$
- $r(t)$: A positive function
- ψ and f : Continuous functions on the real line $\mathbb{R}$, with $\psi(x) > 0$, $\forall x \in \mathbb{R}$ and $yf(y) > 0$ for $y \neq 0$
- g : Continuously differentiable function on the real line, $\mathbb{R}$, except possible at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$ and k is a constant
- H : A continuous function on $[t_0, \infty) \times \mathbb{R}^2$, with $\frac{H(t,y,x)}{g(x)} \leq p(t)$, $\forall t \in [t_0, \infty)$, $y \in \mathbb{R}$ and $x \neq 0$

By a solution of (1), we mean a nontrivial function $x(t)$ satisfying (1). A solution $x(t)$ is said to be oscillatory if it has a sequence of zero clustering at ∞ and non-oscillatory otherwise. Thus a non-oscillatory is either eventually positive or eventually negative. Equation (1) is called oscillatory if all its solutions are oscillatory and otherwise it is called non-oscillatory.

Both oscillatory and non-oscillatory behavior of solutions for various classes of second order differential equations have been widely discussed in the literature, see for example; Elabbasy and Elsharabasy (1997), Grace and Lalli (1980, 1992), Grace (1990), Kiran and Rogovchenko (2001), Philos (1975), Wong (1986), Wong and Yeh (1992), Wintner (1949), Yan (1986) and Yeh (1982). There are a great number of papers addressing particular cases of Eq. (1), such as the following equations (Li, 1998; Manojlovic, 1999, 1991; Tiryakiand Ayanlar, 2004):

$$\left(r(t)x'(t)^{\alpha}\right)' + q(t)g(x(t)) = H(t), \quad (2)$$

$$\left(r(t)\psi(x)x'(t)\right)' + q(t)g(x) = H(t), \quad (3)$$

$$\left(r(t)x'(t)\right)' + Q(t,x) = H(t,x(t),x'(t)), \quad (4)$$

$$\left(r(t)\psi(x(t))f(x'(t))\right)' + q(t)g(x(t)) = H(t). \quad (5)$$

An important tool in the study of oscillatory behaviour of solutions of these equations is the averaging technique which goes back as far as the classical result of Wintner (1949) which states that (2) with $r(t) = 1$, $\alpha = 1$, $g(x) = x$ and $H(t) = 0$ is oscillatory if:

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$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(u)\,du\,ds=\infty.$$

In this study, the comparison between our results and the previously known results are presented and some examples of the main results are illustrated. Furthermore, we expand some of the previous equations, (Baculikova, 2006; Tiryaki and Ayanlar, 2004) as well as the expansion and development of some of the previous conditions (Elabbasy *et al*., 2005; Graef *et al*., 1978; Remili, 2008).

MAIN RESULTS

In this section, we use the Riccati technique to establish sufficient conditions for Eq. (1) to be oscillatory.

Theorem 1: Suppose that:

O_1: $\dfrac{g'(x)}{\psi(x)}\geq K>0$ for a constant K and $x\neq 0$,

O_2: $\displaystyle\int_{\pm\varepsilon}^{\pm\infty}\frac{\psi(y)}{g(y)}dy<\infty$ for all $\varepsilon>0$,

O_3: $0<k_1\leq\dfrac{f(y)}{y}\leq k_2$

for all constants k_1,k_2, $y=x'(t)\neq 0$ and $\lim_{t\to 0}\dfrac{f(y)}{y}$ exist.

There also exists a positive function $\rho\in C^1[t_0,\infty)$ such that $\left(\rho(t)r(t)\right)'\leq 0$ for all $t\geq t_0$, and:

O_4:

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)[q(u)-p(u)]-\frac{(\rho'(u))^2}{4M\rho(u)}r(u)\right]du\,ds=\infty,$$

where,

$$M=\frac{K}{k_2}.$$

Then Eq. (1) is oscillatory.

Proof: On the contrary we assume that Eq. (1) has a non-oscillatory solution, $x(t)$. We suppose without loss of generality that $x(t)>0$ for all $t\in[t_0,\infty)$. We define the function $w(t)$ as:

$$w(t)=\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}\ \text{for all } t\geq t_0. \tag{6}$$

Differentiating Eq. (6) and substituting Eq. (1) imply:

$$w'(t)=\frac{\rho(t)\left[r(t)\psi(x(t))f(x'(t))\right]'}{g(x(t))}+\frac{\rho'(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}-\frac{\rho(t)r(t)\psi(x(t))f(x'(t))g'(x(t))x'(t))}{g^2(x(t))},$$

$$w'(t)=\frac{\rho(t)H(t,x'(t),x(t))}{g(x(t))}-\rho(t)q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{1}{\rho(t)r(t)\psi(x(t))f(x'(t))}x'(t)g'(x(t))w^2(t).$$

From conditions O_1 and O_3 we obtain:

$$w'(t)\leq\rho(t)\left(p(t)-q(t)\right)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{K}{\rho(t)r(t)k_2}w^2(t),$$

$$\rho(t)\left(q(t)-p(t)\right)\leq\frac{\rho'(t)}{\rho(t)}w(t)-\frac{M}{\rho(t)r(t)}w^2(t)-w'(t).$$

Integrating from t_0 to t gives following result:

$$\int_{t_0}^{t}\rho(s)\left(q(s)-p(s)\right)ds\leq w(t_0)-w(t)-\int_{t_0}^{t}\left(\frac{M}{r(s)\rho(s)}w^2(s)-\frac{\rho'(s)}{\rho(s)}w(s)\right)ds,$$

$$\int_{t_0}^{t}\rho(s)\left(q(s)-p(s)\right)-\frac{r(s)\rho(s)}{4M}\left(\frac{\rho'(s)}{\rho(s)}\right)^2ds\leq w(t_0)-w(t)-\int_{t_0}^{t}\left[\sqrt{\frac{M}{r(s)\rho(s)}}w(s)-\frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}}\frac{\rho'(s)}{\rho(s)}\right]^2ds,$$

$$\leq w(t_0)-w(t),$$

or,

$$\int_{t_0}^{t}\left[\rho(s)\left(q(s)-p(s)\right)-\frac{r(s)\rho(s)}{4M}\left(\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds\leq w(t_0)-\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}.$$

Using condition O_3:

$$\int_{t_0}^{t}\left[\rho(s)\left(q(s)-p(s)\right)-\frac{r(s)}{4M}\frac{(\rho'(s))^2}{\rho(s)}\right]ds\leq w(t_0)-k_1\frac{\rho(t)r(t)\psi(x(t))x'(t)}{g(x(t))}.$$

Taking a second integration from t_0 to t we have:

$$\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\left(q(u)-p(u)\right)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]du\,ds\leq w(t_0)(t-t_0)-k_1\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds. \tag{7}$$

Since $r(t)\rho(t)$ is non-increasing, then by the Bonnet's Theorem there exists an $\eta\in[t_0,t]$ such that:

$$-k_1\int_{t_0}^{t}\rho(s)r(s)\frac{\psi(x(s))x'(s)}{g(x(s))}ds=-k_1r(t_0)\rho(t_0)\int_{t_0}^{\eta}\frac{\psi(x(s))x'(s)}{g(x(s))}ds$$
$$=k_1r(t_0)\rho(t_0)\int_{x(\eta)}^{x(t_0)}\frac{\psi(y)}{g(y)}dy$$
$$<\begin{cases}0, & \text{if } x(t_0)<x(\eta),\\ k_1r(t_0)\rho(t_0)\displaystyle\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy & \text{if } x(t_0)>x(\eta),\end{cases}$$

hence,

$$-\infty < -k_1 \int_{t_0}^{t} r(s)\rho(s) \frac{\psi(x)x'(s)}{g(x)} ds < L_1,$$

where,

$$L_1 = k_1 r(t_0)\rho(t_0) \int_{\varepsilon}^{\infty} \frac{\psi(y)}{g(y)} dy.$$

Equation (7) becomes:

$$\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]duds \le w(t_0)(t-t_0)+L_1. \quad (8)$$

Dividing (8) by t and taking the upper limit as $t \to \infty$:

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]du\,ds \le \limsup_{t\to\infty}\frac{1}{t}\left[w(t_0)(t-t_0)+L_1\right] < \infty.$$

This contradicts the assumption O_4, which completes the proof.

Example 1: Consider the following equation:

$$\left[\frac{1}{t}\left(3x'(t)+\frac{(x'(t))^3}{(x'(t))^2+1}\right)\right]'+\left(\frac{1}{2}+\sin t\right)x^3(t)=\frac{x^7(t)\cos t\sin x'(t)}{\left(x^4(t)+1\right)t^3}, \quad t\ge\frac{\pi}{2}.$$

Notice that:

$$\frac{H(t,x'(t),x(t))}{g(x(t))}=\frac{x^7\cos t\sin x'}{(x^4+1)t^3}\times\frac{1}{x^3}\le\frac{1}{t^3}=p(t),\ \forall x'\in\mathbb{R},\ x\in\mathbb{R}\ \text{and}\ t\ge t_0,$$

and

$$\frac{g'(x)}{\psi(x)}=\frac{3x^2}{1}\ge 3=k,$$

$$\int_{\pm\varepsilon}^{\pm\infty}\frac{\psi(y)}{g(y)}dy=-\frac{1}{2y^2}\bigg|_{\pm\varepsilon}^{\pm\infty}=\frac{1}{2\varepsilon^2}<\infty,$$

$$3<3+\frac{(x'(t))^2}{(x'(t))^2+1}<4,\ \forall y\ne 0,\ \lim_{y\to 0}\frac{f(y)}{y}=3.$$

Let,

$$\rho(t)=1\Rightarrow\rho'(t)=0\in C^1\left[t_0,\infty\right),\ \left(\rho(t)r(t)\right)'=-\frac{1}{t^2}<0,$$

O_4: $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]duds$$

$$=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left(\int_{t_0}^{s}1\left[\frac{1}{2}+\sin u-\frac{1}{u^3}\right]-\frac{\frac{1}{u}}{4M}\times 0\,du\right)ds$$

$$=\infty.$$

By Theorem 1, this equation is oscillatory.

Remark 1: Theorem 1 improves and expands to the Theorem 4 of Graef *et al*. (1978) and to the Theorem 1 of Elabbasy *et al*. (2005).

Theorem 2: Assume that O_2 holds and:

O_5: $\frac{f(y)}{y}>L>0$; for a constant L and $y\ne 0$,

there exists a positive continuously differentiable function ρ defined as in Theorem 1 and $\int_{t_0}^{\infty}\rho(s)ds=\infty$ Then Eq. (1) is oscillatory if:

O_6:
$$\limsup_{t\to\infty}\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\rho(s)\left\{\int_{t_0}^{s}\left[q(u)-p(u)\right]du\right\}ds=\infty.$$

Proof: On the contrary we assume that Eq. (1) has a non-oscillatory solution $x(t)$. We suppose without loss of generality that $x(t)>0$ for all $t\in(t_0,\infty)$. We define the function $w(t)$ as:

$$w(t)=\rho(t)\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds\ \text{ for all } t\ge t_0.$$

Thus, for every $t\ge t_0$ we obtain:

$$w'(t)=\rho'(t)\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds+\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}. \quad (9)$$

From Eq. (1) and definition function H, we have:

$$\frac{\left[r(t)\psi(x(t))f(x'(t))\right]'}{g(x(t))}=\frac{H(t,x,x')}{g(x(t))}-q(t)$$

$$\frac{\left[r(t)\psi(x(t))f(x'(t))\right]'}{g(x(t))}\le p(t)-q(t).$$

Integrate the above inequality from t_0 to t and integrate the left hand side by parts we obtain:

$$\frac{\left[r(t)\psi(x(t))f(x'(t))\right]}{g(x(t))}-B+\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))g'(x(s))x'(s)}{g^2(x(s))}ds\le\int_{t_0}^{t}p(s)-q(s)ds,$$

where,

$$B = \frac{r(t_0)\psi(x(t_0))f(x'(t_0))}{g(x(t_0))}.$$

Now, multiply the last inequality by ρ (t) we obtain:

$$\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))} \le B\rho(t) - \rho(t)\int_{t_0}^{t} q(s) - p(s)ds. \quad (10)$$

Substituting (10) in (9) we have:

$$w'(t) - \rho'(t)\int_{t_0}^{t} \frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds \le \rho(t)B - \rho(t)\int_{t_0}^{t}(q(s) - p(s))ds.$$

Integrate from t_0 to t we have:

$$\int_{t_0}^{t}\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\,ds \le w(t_0) - w(t) + B\int_{t_0}^{t}\rho(s)ds + \int_{t_0}^{t}\rho'(s)\int_{t_0}^{s}\frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du\,ds \quad (11)$$

Now evaluate using integration by parts:

$$\int_{t_0}^{t}\rho'(s)\int_{t_0}^{s}\frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du\,ds = \rho(s)\int_{t_0}^{s}\frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du\Bigg|_{t_0}^{t}$$

$$-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds$$

$$= \rho(t)\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds - \int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds$$

$$= w(t) - \int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds. \quad (12)$$

Substituting (12) in (11) we obtain:

$$\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds \le w(t_0) - w(t) + B\int_{t_0}^{t}\rho(s)ds + w(t) - \int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds$$

$$\le w(t_0) + B\int_{t_0}^{t}\rho(s)ds - \int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds.$$

From condition O_5 we get:

$$\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds \le w(t_0) + B\int_{t_0}^{t}\rho(s)ds - L\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds. \quad (13)$$

Since $(\rho(t)r(t))' \le 0$ then by Bonnet's Theorem there exist a $\eta \in [t_0, t]$ such that:

$$-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds = -\rho(t_0)r(t_0)\int_{t_0}^{\eta}\frac{\psi(x(s))x'(s)}{g(x(s))}ds$$

$$= \rho(t_0)r(t_0)\int_{x(\eta)}^{x(t_0)}\frac{\psi(y)}{g(y)}dy$$

$$< \begin{cases} 0 & \text{if } x(t_0) < x(\eta), \\ L\,r(t_0)\rho(t_0)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy & \text{if } x(t_0) > x(\eta), \end{cases}$$

hence,

$$-\infty < -\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds \le B_1,$$

where,

$$B_1 = \rho(t_0)r(t_0)\int_{x(\eta)}^{x(t_0)}\frac{\psi(y)}{g(y)}dy.$$

Consequently:

$$\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds \le M + B\int_{t_0}^{t}\rho(s)ds,$$

where,

$M = w(t_0) + LB_1$

Then,

$$\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds \le \frac{M}{\int_{t_0}^{t}\rho(s)ds} + B.$$

Taking the upper limit as $t \to \infty$ which contradicts the assumption O_6. This completes the proof.

Example 2: Consider the nonlinear differential equation:

$$\left[\frac{1}{t}x^2(t)x'(t)\right]' + t^3x(t)\left(1+x^2(t)\right)^2 = \frac{x^3(t)}{\left(x^2(t)+1\right)^2}\sin t\frac{(x'(t))^2}{(x'(t))^2+1}, \quad t \ge t_0 = 1.$$

Notice that:

$$\frac{H(t,x'(t),x(t))}{g(x(t))} = x^3(t)\sin t\frac{(x'(t))^2}{(x'(t))^2+1}\times\frac{1}{x(t)\left(1+x^2(t)\right)^2}$$

$$\le \sin t = p(t), \quad \forall x' \in \mathbb{R},\ x \in \mathbb{R} \text{ and } t \ge t_0 = 1,$$

$$\int_{\pm\varepsilon}^{\pm\infty}\frac{\psi(y)}{g(y)}dy = \int_{\pm\varepsilon}^{\pm\infty}\frac{y}{(1+y^2)^2}dy = \frac{1}{4\left(1+\varepsilon^2\right)^2} < \infty.$$

Let,

$$r(t) = \frac{1}{t}, \rho(t) = 1$$

We have:

$$\left(\rho(t)r(t)\right)' = -\frac{1}{t^2} < 0$$

and

$$\int_{t_0}^{\infty} \rho(s)ds = \int_{1}^{\infty} ds = \infty,$$

and

$$\limsup_{t\to\infty}\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\rho(s)\int_{t_0}^{s}\left[q(u)-p(u)\right]du\,ds = \limsup_{t\to\infty}\left[\int_{1}^{t}ds\right]^{-1}\int_{1}^{t}\int_{1}^{s}\left[u^3-\sin u\right]du\,ds$$
$$= \limsup_{t\to\infty}\frac{1}{t-1}\left[\frac{t^5}{20}+\sin t-\frac{1}{4}t-t\cos 1-\frac{1}{20}+\sin 1-\frac{1}{4}-\cos 1\right]$$
$$= \infty.$$

Hence, this equation is oscillatory by Theorem 2.

Remark 2: It is straightforward to check that the conclusion of the Theorem 2 is still true if:

$$0 < \int_{t_0}^{\infty} \rho(s)ds < \infty,$$

$$\lim_{t\to\infty}\sup \int_{t_0}^{t} \rho(s)\left\{\int_{t_0}^{s}\left[q(u)-p(u)\right]du\right\}ds = \infty.$$

Remark 3: If $r(t) = 1, \psi\left(x(t)\right) = 1, f\left(x'(t)\right) = x', g\left(x(t)\right) = x$ and $H\left(t, x(t), x'(t)\right) = 0$ then Theorem 2 reduces to Wintner's theorem (1949).

Remark 4: Remili (2008) has established some oscillation results for Eq. (1) with $\psi(x) = 1, f(x'(t)) = x'$. These results require that:

$$a(t) \le a_1$$

and

$$\lim_{t\to\infty}\inf \int_{T}^{t} Z(s)ds > -\lambda\,(\lambda > 0) \text{ for all large } T;$$

$$Z(s) = R(s)\left[q(s)-p(s)\right],$$

which are not required in Theorem 2.

ACKNOWLEDGMENT

This research has been completed with the support of these grants: UKM-DLP-2011-049 and FRGS/ 1/2012/SG04/UKM/01/1.

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