

## Research Article

### Comparison of HPM and PEM for the Flow of a Non-newtonian Fluid between Heated Parallel Plates

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**Abstract:** The present paper studies the heat transfer flow of a third grade fluid between two heated parallel plates for the two models: constant viscosity model and Reynold's model. In each case the nonlinear momentum equation and the energy equation have been solved using HPM and PEM. Graphs for the velocity and temperature profiles are presented and discussed for the various values of parameters entering the problem. The dominant effect is governed by whether or not the fluid is non-Newtonian, the temperature effects being relegated to a less dominant role.

**Keywords:** Constant viscosity model, homotopy perturbation method, momentum equation, PEM, Reynold's model, third grade fluid

## INTRODUCTION

In general, the flowing mixtures consist of solid particles in a fluid such as coal based slurries exhibit non-Newtonian characteristics. These mixtures are important in a variety of industrial applications and heat transfer plays an important role in handling and processing of these mixtures. There are properties of fluid behavior which cannot be explained on the basis of the classical, linearly viscous models. Several constitutive equations have been suggested to characterize such non-Newtonian behavior. Amongst these are fluid of the different type of grade  $n$  (Truesdell and Noll, 1965), the incompressible and homogeneous fluid of grade 1 being the linearly viscous Newtonian fluid. For example, it has been shown that the substantial performance benefits can be obtained if coal-water mixture is pre-heated (Massoudi and Christie, 1995; Tsai *et al.*, 1988).

In this study, we consider the model described by Szeri and Rajagopal (1985) for heat transfer flow of a third grade fluid between two parallel plates maintained at different temperature located in the  $y = 0$  and  $y = h$  planes, respectively, of an orthogonal cartesian coordinate system. Szeri and Rajagopal (1985) examined this model for the two cases via a similarity transformation and concluded that the temperature dependence was not important for third grade fluid for the considered parameters and the variable viscosity solutions were not too distinct from that of constant velocity. With a slight modification, we consider this

model and compare our results produced by PEM and HPM with those produced by Szeri and Rajagopal (1985). In each case the nonlinear momentum equation and the energy equation have been solved using HPM and PEM. Graphs for the velocity and temperature profiles are presented and discussed for the various values of parameters entering the problem.

There have been several studies involving heat and transfer in the non-Newtonian fluids, but most of them seem to lack a systematic and rational treatment of the thermodynamics of the problem: while the stress constitutive equation is altered to account for non-Newtonian behavior, the constitutive equation for the specific Helmholtz free energy or the heat flux vector are left unchanged. Although this might be correct for a particular fluid, it does not seem proper to assume the same a priori.

In this study, we attempt a thermodynamically consistent study of the heat transfer problem under consideration. We consider two models in our approach: constant viscosity model and Reynolds model (Szeri and Rajagopal, 1985). The governing differential equations for the velocity and temperature are non-linear whose exact solutions are not available. Therefore, asymptotic methods prove a powerful tool to obtain approximate solution of these equations. An excellent review of some of these methods is given in (Siddiqui *et al.*, 2008).

Of various asymptotic methods, the one is the homotopy perturbation method provided by He (1998a,

1999, 2000, 2003) which is coupling of the traditional perturbation method and the homotopy concept used in the topology. In a series of papers by Abbasbandi (2006) and He (2004a, b, 2005a, 1998b, 2005b) and several others have not only applied this method successfully to obtain the solution of currently important problems in science and technology, but have also shown its effectiveness and reliability. Siddiqui *et al.* (2006a, b, 2007, 2008) have applied this method to analyze flow problems of non-Newtonian fluid mechanics. In this study we have used HPM and PEM to solve the problem. We have also given a comparison of different perturbation techniques with illustrative examples of Shakil *et al.* (2013) and Wahab *et al.* (2013a, b, c).

### METHODOLOGY

**Basic equations:** The basic laws of the conservation of mass, conservation of momentum and the conservation of energy for an incompressible fluid are given by:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \underline{f} + \text{div} \mathbf{T}, \tag{2}$$

$$\rho \frac{d\varepsilon}{dt} = \mathbf{T} \cdot \mathbf{L} - \text{div} \mathbf{q} + \rho r, \tag{3}$$

where,

$\mathbf{v}$  = The velocity field

$\underline{f}$  = The body force

$\mathbf{T}$  = The stress tensor

$\rho$  = The constant density

$\varepsilon$  = The internal energy

$r$  = The radiant energy

The constitutive equation for a third grade fluid is:

$$\mathbf{T} = -p\mathbf{I} + \mu(\theta)\mathbf{A}_1 + \alpha_1(\theta)\mathbf{A}_2 + \alpha_2(\theta)\mathbf{A}_1^2 + \beta_1(\theta)\mathbf{A}_3 + \beta_2(\theta)[\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1] + \beta_3(\theta)(\text{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{4}$$

Here  $-p\mathbf{I}$  denotes the indeterminate part of stress due to the constraint of incompressibility,  $\mu(\theta)$  is the coefficient of viscosity and  $\alpha_1(\theta)$ ,  $\alpha_2(\theta)$  are material moduli, usually referred as normal stress coefficients. The Rivlin-Ericksen tensor,  $\mathbf{A}_n$  are defined as:  $\mathbf{A}_0 = \mathbf{I}$ , the identity tensor and:

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\nabla\mathbf{v}) + \mathbf{A}_{n-1}(\nabla\mathbf{v})^T. \quad n \geq 1 \tag{5}$$

In our analysis, we assume that the fluid is thermodynamically compatible; hence the stress constitutive relation reduces to:

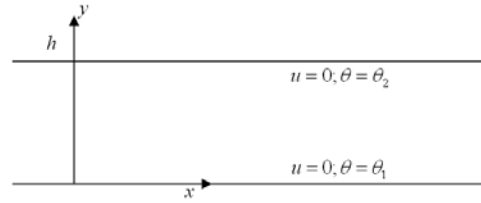


Fig. 1: The  $x$  coordinate direction that is chosen parallel the external pressure gradient

$$\mathbf{T} = -p\mathbf{I} + \mu(\theta)\mathbf{A}_1 + \alpha_1(\theta)\mathbf{A}_2 + \alpha_2(\theta)\mathbf{A}_1^2 + \beta(\theta)(\text{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{6}$$

**Governing equations:** With reference to the model by Szeri and Rajagopal (1985), consider Poiseuille flow of a thermodynamically compatible third grade fluid between parallel plates located at  $y = 0$  and  $y = h$ , respectively (Fig. 1), such that we seek velocity field of the form:

$$\mathbf{v} = [u(y), 0, 0], \tag{7}$$

In the absence of body forces, the balance of linear momentum:

$$\text{div} \mathbf{T} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}. \tag{8}$$

Using velocity profile (7), Eq. (8) reduces to:

$$\frac{d}{dy} \left[ \mu(\theta) \frac{du}{dy} \right] + 2 \frac{d}{dy} \left[ \beta(\theta) \left( \frac{du}{dy} \right)^3 \right] = \frac{\partial p}{\partial x}, \tag{9}$$

$$\frac{d}{dy} \left\{ [2\alpha_1(\theta) + \alpha_2(\theta)] \left( \frac{du}{dy} \right)^2 \right\} = \frac{\partial p}{\partial y}, \tag{10}$$

$$0 = \frac{\partial p}{\partial z}. \tag{11}$$

Since the fluid is incompressible and is hence constrained to satisfy:

$$\text{div} \mathbf{V} = 0. \tag{12}$$

The modified pressure  $P$  is defined through:

$$P = p - [2\alpha_1(\theta) + \alpha_2(\theta)] \left( \frac{du}{dy} \right)^2. \tag{13}$$

Equation (9) to (11) take the simpler form:

$$\frac{d}{dy} \left[ \mu(\theta) \frac{du}{dy} \right] + 2 \frac{d}{dy} \left[ \beta_3(\theta) \left( \frac{du}{dy} \right)^3 \right] = \frac{\partial P}{\partial x} \tag{14}$$

as  $P = P(x)$  only.

The appropriate form of the energy equation produced as follows:

$$\rho \frac{d\varepsilon}{dt} = \mathbf{T} \cdot \mathbf{L} - \text{div} \mathbf{q} + \rho r, \quad (15)$$

$$\frac{d\varepsilon}{dt} = 0. \quad (24)$$

where,  $\varepsilon$  denotes the internal energy and  $r$  is the radiant energy, both per unit mass and:

$$\begin{aligned} \mathbf{T} \cdot \mathbf{L} = & \frac{1}{2} \mu(\theta) |A_1|^2 + \frac{1}{4} \alpha_1(\theta) \frac{d}{dt} |A_1|^2 \\ & + \frac{\alpha_1(\theta) + \alpha_2(\theta)}{2} \text{tr} \mathbf{A}_1^3 + \frac{1}{2} \beta_3(\theta) |A_1|^4, \end{aligned} \quad (16)$$

Substituting (19) and (21) into (15), in the absence of radial heating, the balance of energy implies that:

$$\mu(\theta) \left(\frac{du}{dy}\right)^2 + 2\beta_3 \theta \left(\frac{du}{dy}\right)^4 + k \frac{d^2\theta}{dy^2} = 0. \quad (25)$$

Which after simplification gives:

$$\mathbf{T} \cdot \mathbf{L} = \mu(\theta) \left(\frac{du}{dy}\right)^2 + 2\beta_3(\theta) \left(\frac{du}{dy}\right)^4. \quad (17)$$

The following variables and parameters introduced to non-dimensionalize (14) and (25):

$$\begin{aligned} \bar{y} &= \frac{y}{h}, \quad \bar{u} = \frac{u}{V}, \\ \bar{T} &= \frac{\theta - \theta_1}{\theta_2 - \theta_1}, \quad \bar{\mu} = \frac{\mu}{\mu_*}, \\ A &= \frac{\beta_* V^2}{\mu_* h^2}, \quad \Gamma = \frac{\mu_* V^2}{k(\theta_2 - \theta_1)} \end{aligned}$$

The heat flux vector  $q$  is given by Fourier's law:

$$q = -k \text{ grad} \theta \quad (18)$$

where,  $k = k(\theta)$  denotes the thermal conductivity, but specify that  $k$  is constant. We now consider the role of internal energy to the energy equation. Since the internal energy is related to the specific Helmholtz free energy through:

$$\varepsilon = \psi + \theta \eta, \quad (19)$$

where,  $\mu_* = \mu(\theta_*)$ ;  $\beta_* = \beta_3(\theta_*)$  and the characteristic velocity is given by  $V = -\frac{h^2 dp}{2\mu dx}$ , but with  $\mu_*$  replacing  $\mu$ ,  $\theta_1 = \theta(0)$  and  $\theta_2 = \theta(h)$  are the wall temperature at the lower and the upper plate, respectively.

The non-dimensional form of Eq. (14) and (25) are:

where  $\eta$  is the entropy, leads to:

$$\varepsilon = \psi(\theta, 0) + \frac{\alpha_1(\theta)}{4\rho} |A_1|^2 + \theta \eta. \quad (20)$$

$$\bar{u}'' \left[ 1 + 6A\bar{\mu}(\bar{u}')^2 \right] = -2 \quad (26)$$

$$\bar{T}'' + \Gamma(\bar{u}')^2 \left[ \bar{\mu} + 2A(\bar{u}')^2 \right] = 0. \quad (27)$$

Equation (26) and (27) can be solved subjected to the boundary conditions:

For the thermodynamically compatible fluid, the specific entropy is related to the specific Helmholtz free energy through:

$$\eta = -\psi_{\theta}, \quad (21)$$

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \quad (28)$$

$$\bar{T}(0) = 0, \bar{T}(1) = 1. \quad (29)$$

where, the suffix denotes differentiation with respect to that variable. It follows from (20) and (21) that:

$$\varepsilon = \psi(\theta, 0) + \frac{\alpha_1(\theta)}{4\rho} |A_1|^2 - \theta \psi_{\theta}. \quad (22)$$

The viscosity is temperature dependent in (26) and (27) and the precise form of the equation of motion depends on the viscosity model chosen to represent the fluid. We refer here two viscosity models and the corresponding form of the non-dimensional equation of motion:

**Constant viscosity model:**

The material properties of the fluid are allowed to be temperature dependent, but require the temperature to satisfy the constraint:

$$\bar{\mu}(\bar{T}) = 1 \quad (30)$$

$$\theta = \theta(y), \quad (23)$$

$$\bar{u}'' \left[ 1 + 6A(\bar{u}')^2 \right] = -2 \quad (31)$$

Then by the virtue of (22) and (23):

Subjected to the boundary conditions:

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \tag{32}$$

**Reynolds's model:**

$$\mu(\theta) = \mu_0 \exp(-m\theta) \tag{33}$$

so that:

$$\bar{\mu}(\bar{T}) = \exp(-M\bar{T}) \tag{34}$$

$$\bar{u}'' \left[ 1 + 6A \frac{(\bar{u}')^2}{\bar{\mu}} \right] - M\bar{T}\bar{u}' = -\frac{2}{\bar{\mu}} \tag{35}$$

where,  $M = m(\theta_2 - \theta_1)$

Subjected to the boundary conditions:

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \tag{36}$$

Our aim here is to compare the results of this model (Szeri and Rajagopal, 1985) with the results produced by HPM and PEM.

**Constant viscosity model by HPM:** In this case, we have two non-linear equations:

$$\bar{T}'' + \Gamma(\bar{u}')^2 [\mu + 2A(\bar{u}')^2] = 0, \tag{37}$$

$$\bar{u}'' [1 + 6A(\bar{u}')^2] = -2, \tag{38}$$

Subjected to the boundary conditions:

$$\bar{T}(0) = 0, \bar{T}(1) = 1 \tag{39}$$

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \tag{40}$$

We have to solve these two equations simultaneously. By applying the basic definition of HPM and expanding  $\bar{T}$  and  $\bar{u}$  as:

$$\bar{T} = T_0 + pT_1 + p^2T_2 + \dots \tag{41}$$

$$\bar{u} = u_0 + pu_1 + p^2u_2 + \dots \tag{42}$$

We will get system of linear equations for temperature distribution as:

$$p^0 T_0'' = 0, \tag{43}$$

Subjected to the boundary conditions:

$$T_0(0) = 0, T_0(1) = 1. \tag{44}$$

$$p^1 T_1'' + \Gamma u_0'^2 + 2A\Gamma u_0'^4 = 0, \tag{45}$$

Subjected to the boundary conditions:

$$T_1(0) = 0, T_1(1) = 0. \tag{46}$$

$$p^2 T_2'' + 2\Gamma u_1' u_0' + 8A\Gamma u_1' u_0'^3 = 0, \tag{47}$$

Subjected to the boundary conditions:

$$T_2(0) = 0, T_2(1) = 0. \tag{48}$$

And the system of linear equations for velocity profile:

$$p^0 u_0'' = -2, \tag{49}$$

Subjected to the boundary conditions:

$$u_0(0) = 0, u_0(1) = 0. \tag{50}$$

$$p^1 u_1'' + 6A u_0'^2 u_0'' = 0, \tag{51}$$

Subjected to the boundary conditions:

$$u_1(0) = 0, u_1(1) = 0. \tag{52}$$

$$p^2 u_2'' + 12A\Gamma u_1' u_0' u_0'' + 6A u_1'' u_0'^2 = 0, \tag{53}$$

Subjected to the boundary conditions:

$$u_2(0) = 0, u_2(1) = 0. \tag{54}$$

Solving these systems of linear equations we get the temperature distribution and velocity profile:

$$\bar{T} = y - \Gamma \left( \frac{y^4}{3} - \frac{2y^3}{3} + \frac{y^2}{2} - \frac{y}{6} \right) - 2A\Gamma \left( \frac{8y^6}{15} - \frac{8y^5}{5} + 2y^4 - \frac{4y^3}{3} + \frac{y^2}{2} - \frac{y}{10} \right). \tag{55}$$

$$\bar{u} = y - y^2 + A(8y^3 - 4y^4 - 6y^2 + 2y). \tag{56}$$

**Constant viscosity model by PEM:** In this case, we have two non-linear equations:

$$\bar{T}'' + \Gamma(\bar{u}')^2 [\mu + 2A(\bar{u}')^2] = 0, \tag{57}$$

$$\bar{u}'' [1 + 6A(\bar{u}')^2] = -2, \tag{58}$$

Subjected to the boundary conditions:

$$\bar{T}(0) = 0, \bar{T}(1) = 1 \tag{59}$$

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \tag{60}$$

We have to solve these two equations simultaneously. Here we have to expand the parameters as:

$$\bar{T} = T_0 + pT_1 + p^2T_2 + \dots \quad (61)$$

$$\bar{u} = u_0 + pu_1 + p^2u_2 + \dots \quad (62)$$

$$\Gamma = p\Gamma_1 + p^2\Gamma_2 + \dots \quad (63)$$

$$A = pA_1 + p^2A_2 + \dots \quad (64)$$

Using these expanded parameters in above equations, we will get system of linear Equations for temperature distribution as:

$$p^0 T_0'' = 0, \quad (65)$$

Subjected to the boundary conditions:

$$T_0(0) = 0, T_0(1) = 1. \quad (66)$$

$$p^1 T_1'' + \Gamma_1 u_0'^2 = 0, \quad (67)$$

Subjected to the boundary conditions:

$$T_1(0) = 0, T_1(1) = 0. \quad (68)$$

$$p^2 T_2'' + 2\Gamma_1 u_1' u_0' + \Gamma_2 u_0'^2 + 2A_1 \Gamma_1 u_0'^4 = 0, \quad (69)$$

Subjected to the boundary conditions:

$$T_2(0) = 0, T_2(1) = 0. \quad (70)$$

And the system of linear equations for velocity profile:

$$p^0 u_0'' = -2, \quad (71)$$

Subjected to the boundary conditions:

$$u_0(0) = 0, u_0(1) = 0. \quad (72)$$

$$p^1 u_1'' + 6A_1 u_0'^2 u_0'' = 0, \quad (73)$$

Subjected to the boundary conditions:

$$u_1(0) = 0, u_1(1) = 0. \quad (74)$$

$$p^2 u_2'' + 12A_1 u_1' u_0' u_0'' + 6A_1 u_1'' u_0'^2 + 6A_2 u_0'' u_0'^2 = 0, \quad (75)$$

Subjected to the boundary conditions:

$$u_2(0) = 0, u_2(1) = 0. \quad (76)$$

Solving these system of linear equations we get the temperature distribution and velocity profile:

$$\begin{aligned} \bar{T} = & y - \Gamma_1 \left( \frac{y^4}{3} - \frac{2y^3}{3} + \frac{y^2}{2} - \frac{y}{6} \right) \\ & - 2A\Gamma_1 \left( -\frac{8y^6}{15} + \frac{8y^5}{5} - 2y^4 + \frac{4y^3}{3} - \frac{y^2}{2} + \frac{y}{10} \right) \\ & - \Gamma_2 \left( \frac{y^4}{3} - \frac{2y^3}{3} + \frac{y^2}{2} - \frac{y}{6} \right). \end{aligned} \quad (77)$$

$$\bar{u} = y - y^2 + A(-8y^3 + 4y^4 + 6y^2 - 2y). \quad (78)$$

**Reynolds's model by HPM:** In this case, we have two non-linear equations:

$$\bar{T}'' + \Gamma(\bar{u}')^2 [\exp(-M\bar{T}) + 2A(\bar{u}')^4] = 0, \quad (79)$$

$$\bar{u}'' + 6A(\bar{u}')^2 \exp(M\bar{T}) - M\bar{T}\bar{u}' = -2\exp(-M\bar{T}) \quad (80)$$

where,  $M = m(\theta_2 - \theta_1)$

Subjected to the boundary conditions:

$$\bar{T}(0) = 0, \bar{T}(1) = 1 \quad (81)$$

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \quad (82)$$

We have to solve these two equations simultaneously. By applying the same procedure as for constant viscosity model, we will get system of linear equations for the temperature distribution as:

$$p^0 T_0'' = 0, \quad (83)$$

Subjected to the boundary conditions:

$$T_0(0) = 0, T_0(1) = 1. \quad (84)$$

$$p^1 T_1'' + u_0'^2 - Mu_0'^2 T_0 + \frac{M^2}{2} u_0'^2 T_0^2 + 2A\Gamma u_0'^4 = 0, \quad (85)$$

Subjected to the boundary conditions:

$$T_1(0) = 0, T_1(1) = 0. \quad (86)$$

And the system of the linear equations for velocity profile:

$$p^0 u_0'' = 0, \quad (87)$$

Subjected to the boundary conditions:

$$u_0(0) = 0, u_0(1) = 0. \quad (88)$$

$$p^1 u_1'' + 6A \left[ u_0'^2 u_0'' + M u_0'^2 u_0'' T_0 + \frac{M^2}{2} u_0'^2 u_0'' T_0^2 - M T_0' u_0' - 2M T_0 + M^2 T_0^2 + 2 \right] = 0, \tag{89}$$

Subjected to the boundary conditions:

$$u_1(0) = 0, u_1(1) = 0. \tag{90}$$

Solving these system of linear equations we get the temperature distribution and velocity profile:

$$\begin{aligned} \bar{T} = & y - \frac{A\Gamma M^4 y^4}{720} - \frac{A\Gamma M^3 y^9}{36} + \Gamma M^2 y^8 \left\{ M^2 \left( \frac{1}{448} + \frac{A}{336} \right) + A \left( \frac{M}{56} - \frac{3}{14} \right) \right\} \\ & + \Gamma M y^7 \left\{ M^2 \left( \frac{A}{21} - \frac{1}{56} \right) + A \left( \frac{2M}{7} - \frac{16}{21} \right) \right\} + \Gamma M y^6 \left\{ -\frac{AM^3}{360} - \frac{M}{30} \left( \frac{1}{40} + A\Gamma \right) + \frac{A}{5} \left( 8 - \frac{16}{3M} \right) \right\} \\ & + \Gamma M y^5 \left\{ \frac{M^2}{30} \left( \frac{1}{4} - A \right) + \frac{M}{5} \left( -2A + \frac{1}{4} \right) + A \left( \frac{16}{5M} - \frac{2}{3} \right) + \frac{1}{10} \right\} \\ & + \Gamma M y^4 \left\{ M^3 \left( \frac{A}{648} - \frac{1}{864} \right) + \frac{M^2}{3} \left( \frac{A}{10} + \frac{1}{24} \right) + \frac{M}{2} \left( -\frac{1}{6} + \frac{A}{9} \right) + A \left( \frac{M}{18} - 1 \right) - \frac{1}{4} \right\} \\ & + \Gamma M y^3 \left\{ \frac{M^2}{3} \left( \frac{1}{72} + \frac{A}{27} \right) + \frac{M}{9} \left( 2A + \frac{1}{2} \right) + \frac{10}{3M} + \frac{4A}{3} + \frac{5}{18} \right\} \\ & + \Gamma M y^2 \left\{ -\frac{AM^3}{1296} - \frac{AM^2}{54} - \frac{M}{6} \left( A + \frac{1}{36} \right) - \frac{2A}{3} - \frac{1}{6} - \frac{3}{2M} \right\}. \end{aligned} \tag{91}$$

$$\begin{aligned} \bar{u} = & \frac{A\Gamma M^5 y^9}{96} + \frac{9AM^4 y^8}{56} + \frac{AM^3 y^7}{14} \left( -\frac{M^2}{6} + \frac{27}{2} - M \right) + \frac{AM^3 y^7}{14} \left( -\frac{M^2}{6} + \frac{27}{2} - M \right) \\ & + \frac{AM^2 y^6}{5} \left( -\frac{2M^2}{3} + \frac{29}{2} - 4M \right) + \frac{AM^2 y^6}{5} \left( -\frac{2M^2}{3} + \frac{29}{2} - 4M \right) + \frac{AM y^5}{5} \left( \frac{M^4}{48} + \frac{M^3}{4} - 2M^2 - \frac{33M}{2} + 24 \right) \\ & + AM y^4 \left( \frac{M^3}{36} + \frac{M^2}{3} - \frac{M}{6} + \frac{4}{M} - \frac{M}{8A} - 7 \right) \\ & + AM y^3 \left( \frac{M^2}{12} + M - \frac{8}{M} - \frac{1}{2A} + \frac{5}{3} \right) + AM y^2 \left( \frac{M}{12A} + \frac{M}{6} + \frac{6}{M} - \frac{1}{MA} + \frac{1}{2A} + 2 \right) \\ & + y \left( -\frac{3AM^5}{1120} - \frac{17AM^4}{504} - \frac{19AM^3}{105} - \frac{3AM^2}{5} + \frac{M^2}{24} - \frac{22AM}{15} - 2A + 1 \right). \end{aligned} \tag{92}$$

**Reynolds's model by PEM:** In this case, we have two non-linear equations:

$$\bar{T}'' + \Gamma (\bar{u}')^2 \left[ \exp(-M\bar{T}) + 2A(\bar{u}')^4 \right] = 0, \tag{93}$$

$$\bar{u}'' + 6A(\bar{u}')^2 \exp(M\bar{T}) - M\bar{T}\bar{u}' = -2 \exp(-M\bar{T}) \tag{94}$$

where  $M = m(\theta_2 - \theta_1)$

Subjected to the boundary conditions:

$$\bar{T}(0) = 0, \bar{T}(1) = 1 \tag{95}$$

$$\bar{u}(0) = 0, \bar{u}(1) = 0. \tag{96}$$

Applying the same procedure as for the constant viscosity model, we will get system of linear equations for temperature distribution as:

$$p^0 T_0'' = 0, \tag{97}$$

Subjected to the boundary conditions:

$$T_0(0) = 0, T_0(1) = 1. \tag{98}$$

$$p^1 T_1'' + \Gamma_1 u_0'^2 = 0, \tag{99}$$

Subjected to the boundary conditions:

$$T_1(0) = 0, T_1(1) = 0. \tag{100}$$

$$p^2 T_2'' + \Gamma_2 u_0'^2 + 2\Gamma_1 u_0' u_1' - M_1 \Gamma_1 u_0'^2 T_0 + 2A_1 \Gamma_1 u_0'^4 = 0, \tag{101}$$

Subjected to the boundary conditions:

$$T_2(0) = 0, T_2(1) = 0. \tag{102}$$

And the system of linear equations for the velocity profile:

$$p^0 u_0'' = -2, \tag{103}$$

Subjected to the boundary conditions:

$$u_0(0) = 0, u_0(1) = 0. \tag{104}$$

$$p^1 u_1'' + 6u_0'^2 - M_1 T_0 u_0 + 2M_1 T_0 = 0, \tag{105}$$

Subjected to the boundary conditions:

$$u_1(0) = 0, u_1(1) = 0. \tag{106}$$

Solving these system of linear equations we get the temperature distribution and velocity profile:

$$\begin{aligned} \bar{T} = & y + \Gamma_1 \left[ -\frac{y^4}{3} + \frac{2y^3}{3} - \frac{y^2}{2} + \frac{y}{6} \right] - \Gamma_2 \left[ \frac{y^4}{3} - \frac{2y^3}{3} + \frac{y^2}{2} - \frac{y}{6} \right] - 2\Gamma_1 \left[ \frac{y^2}{2} - \frac{4y^3}{3} + 2y^4 + \frac{8y^5}{5} + \frac{8y^6}{15} - \frac{y}{10} \right] \\ & - M_1 \left[ \frac{y^7}{84} - \frac{11y^6}{360} + \frac{7y^5}{60} - \frac{y^4}{12} - \frac{y^3}{10} + \frac{3y^2}{20} - \frac{163y}{2520} \right] - \Gamma_1 M_1 \left[ \frac{y^3}{6} - \frac{y^4}{3} + \frac{y^5}{5} - \frac{y}{30} \right] \\ & - 2A_1 \Gamma_1 \left[ \frac{y^2}{2} - \frac{4y^3}{3} + 2y^4 - \frac{8y^5}{5} + \frac{8y^6}{15} - \frac{y}{10} \right]. \end{aligned} \tag{107}$$

$$\bar{u} = y - y^2 - [3y^3 + 2y^4 - 4y^3 - y] + M_1 \left[ \frac{y^5}{20} + \frac{y^4}{12} - \frac{y^3}{3} + \frac{3y}{10} \right]. \tag{108}$$

### DISCUSSION OF RESULTS

In both the cases: Constant viscosity model and Reynold's model, we obtained velocity profiles and temperature distributions for different values of M and A keeping the  $\gamma$  fixed. For dimensionless velocity and temperature distributions for Constant viscosity model and Reynold's model by HPM keeping  $\gamma = 10$ , it is clear from the Fig. 2 that for  $M = 0, 1$  and  $5$ , departure from symmetry is slight. To investigate the effects of M on the temperature

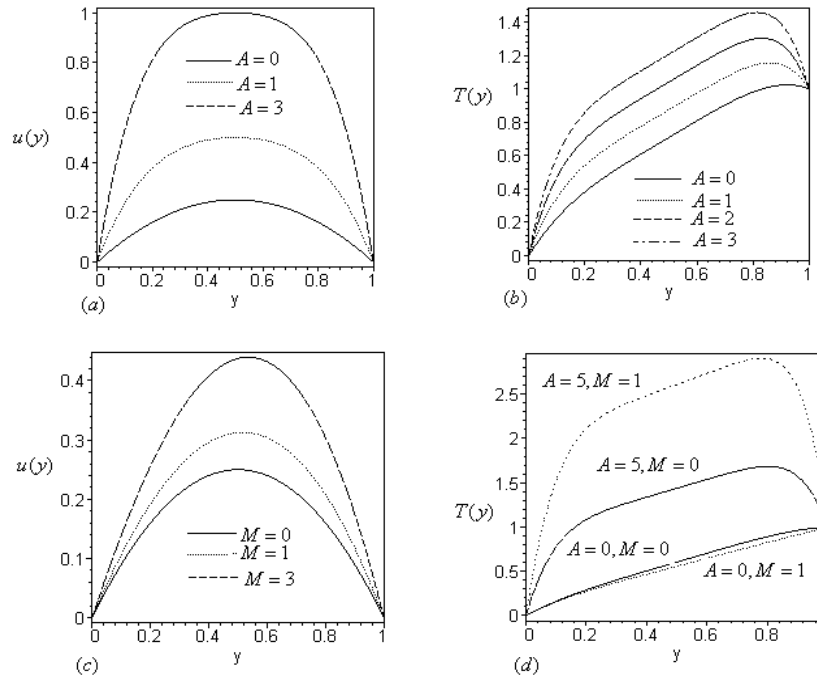


Fig. 2: Dimensionless velocity and temperature distributions for constant viscosity model and Reynolds's model by HPM keeping  $\gamma = 10$

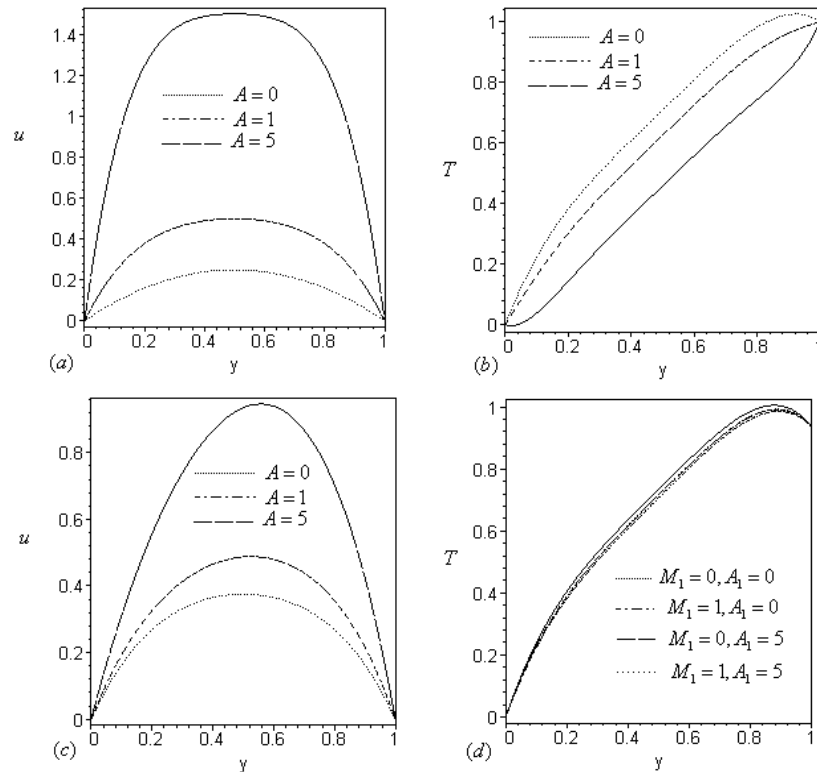


Fig. 3: Dimensionless velocity and temperature distributions for constant viscosity model and Reynolds's model by PEM keeping  $\gamma_1 = 0$  and  $\gamma_2 = 10$

distribution, we include viscous heating  $\gamma = 10$ . At this moderate rate of viscous heating, the temperature shows strong dependence on  $M$  for the Newtonian fluid only.

For the non-Newtonian fluid, the temperature profiles obtained at different values of  $M$  coalesce almost into single curve. For dimensionless velocity



and temperature distributions for Constant viscosity model and Reynolds's model by PEM keeping  $\gamma_1 = 0$  and  $\gamma_2 = 10$ , Fig. 3 indicates that at this moderate rate of viscous heating the temperature shows strong dependence on  $M$  for the Newtonian fluid only. For the non-Newtonian fluid the temperature profiles obtained at different values of  $M$  coalesce almost into a single curve.

It is clear from both the figures that the temperature and velocity distributions remain sensibly invariant with respect to the viscosity index  $M$  in non-Newtonian fluids if the viscosity-temperature law for these fluids is given by Reynolds' formula. The Fig. 3 indicates that even slight departure from Newtonian behavior places the fluid in the regime where dependence on the velocity index is only slight.

### CONCLUSION

Here we observed the flow of third grade fluid between heated parallel plates using HPM and PEM. It is clear from the figures that results obtained by HPM (Fig. 2) and PEM (Fig. 3) are very close to the numerical results obtained by Szeri and Rajagopal (1985). Thus, an important conclusion is that the dominant effect is governed by whether or not the fluid is non-Newtonian, the temperature effects being relegated to a less dominant role. We conclude that from point of view of velocity and temperature distribution in poiseuille flow, the temperature dependence is not important for the third grade fluid for the considered parameters and the variable viscosity solutions were not too distinct from that of constant velocity even if the fluid is slightly non-Newtonian.

### REFERENCES

- Abbasbandi, S., 2006. Application of He's homotopy perturbation method for laplace transform. *Chaos Soliton. Fract.*, 30(5): 1206-1212.
- He, J.H., 1998a. An approximation solution technique depending upon an artificial parameter: A special example. *Commun. Nonlinear Sci.*, 3(2): 92-97.
- He, J.H., 1998b. Newton-like iteration method for solving algebraic equations. *Commun. Nonlinear Sci.*, 3(2): 106-109.
- He, J.H., 1999. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Eng.*, 178(3-4): 257-262.
- He, J.H., 2000. A coupling method of a homotopy technique and perturbation technique for non-linear problems. *Int. J. Nonlinear Mech.*, 35(1): 37-43.
- He, J.H., 2003. Homotopy perturbation method: A new nonlinear analytical technique. *Appl. Math. Comput.*, 135(1): 73-79.
- He, J.H., 2004a. Asymptotology by homotopy perturbation method. *Appl. Math. Comput.*, 15(3): 591-596.
- He, J.H., 2004b. The homotopy perturbation method for nonlinear oscillators and discontinuities. *Appl. Math. Comput.*, 151(1): 287-292.
- He, J.H., 2005a. Application of homotopy perturbation method to nonlinear wave equations. *Chaos Soliton. Fract.*, 26(3): 695-700.
- He, J.H., 2005b. Homotopy perturbation method for bifurcation of nonlinear problems. *Int. J. Nonlinear Sci.*, 6(2): 207.
- Massoudi, M. and I. Christie, 1995. Effects of variable Dissipation on the flow of a third grade fluid in a pipe. *Int. J. Nonlinear Mech.*, 30(5): 687-699.
- Shakil, M., T. Khan, H.A. Wahab and S. Bhatti, 2013. A comparison of Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) for nonlinear problems. *IMPACT: Int. J. Res. Appl. Nat. Soc. Sci.*, 1(3): 37-48.
- Siddiqui, A.M., M. Ahmed and Q.K. Ghori, 2006a. Couette and Poiseuille flows for non-newtonian fluids. *Int. J. Nonlinear Sci.*, 7(1): 15.
- Siddiqui, A.M., R. Mahmood and Q.K. Ghori, 2006b. Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder. *Phys. Lett.*, 352(4-5): 404-410.
- Siddiqui, A.M., M. Ahmed and Q.K. Ghori, 2007. Thin film flow of non-Newtonian fluids on a moving belt. *Chaos Soliton. Fract.*, 33(3): 1006-1016.
- Siddiqui, A.M., R. Mahmood and Q.K. Ghori, 2008. Homotopy perturbation method for thin film flow of a third grade fluid down an inclined plane. *Chaos Soliton. Fract.*, 35(1): 140-147.
- Szeri, A.Z. and K.R. Rajagopal, 1985. Low of non-Newtonian fluid between heated parallel plates. *Int. J. Nonlinear Mech.*, 20(2): 91-101.
- Truesdell, C. and W. Noll, 1965. The Non-linear Field Theories of Mechanics. 2nd Edn., In: Flugge (Ed.), *Handbuch der Physik* (German). Springer-Verlag, Berlin.
- Tsai, C.Y, M. Novack and G. Roffle, 1988. Rheological and heat transfer characteristics of flowing coal-water mixtures. Final Report, DOE/ MC/23255-2763.
- Wahab, H.A., M. Shakil, T. Khan, S. Bhatti and M. Naeem, 2013a. A comparative study of a system of Lotka-Volterra type of PDEs through perturbation methods. *Computat. Ecol. Software*, 3(4): 110-125.
- Wahab, H.A., T. Khan, M. Shakil, S. Bhatti and M. Naeem, 2013b. Analytical treatment of system of KdV equations by Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM). *Comput. Ecol. Software*, 4(1): 63-81.
- Wahab, H.A., S. Bhatti, M. Naeem, M.T. Qureshi and M. Afzal, 2013c. A mathematical model for the rods with heat generation and convective cooling. *J. Basic Appl. Scientific Res.*, ISSN: 2090-4304 (Print), ISSN: 2090-424x (Online), (Accepted).