

Research Article

Solution of the System of Fifth Order Boundary Value Problem using Quartic Spline

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Abstract: The aim of the study is the solution of the system of fifth order boundary value problems associated with obstacle, unilateral and contact problems using quartic spline. These problems arise in several branches of pure and applied sciences and in engineering including transportation, equilibrium, optimization, mechanics, structural analysis, fluid flow through porous media and image processing in the medical sciences. The results are compared with the exact solution. Two examples are considered for the numerical illustration of the method developed and the results are encouraging.

Keywords: Contact problems, obstacle problems, quartic spline, system of boundary value problems, variational inequalities

INTRODUCTION

The contact, unilateral, obstacle and equilibrium problems arising in different branches of pure and applied sciences can be studied using the notion of variational inequalities. During the past few years, this has emerged as an interesting and important branch of applied mathematics. The general variational inequalities can be characterized by a system of differential equations using the penalty function technique, if the obstacle function is known. This technique was used by Lewy and Stampacchia (1969) to study the regularity of the solution of variational inequalities. The main advantage of this technique is its simple applicability in solving obstacle and unilateral problems.

Al-Said (1996) developed the solution of system of second order boundary value problems using quadratic spline. Gao and Chi (2006) solved a system of third-order boundary value problems associated with third-order obstacle problems using the quartic B-splines and the method is claimed to be of second order. Islam *et al.* (2005) developed the solution of a system of third-order boundary value problems using non polynomial spline and the method is claimed to be of second order as well.

Siddiqi and Akram (2007a to c) solved the system of fourth order boundary value problem using cubic nonpolynomial spline, cubic spline and nonpolynomial spline. Noor *et al.* (2011) solved fifth order obstacle problem using variation of parameters method, but the exact solution of the problem, given in the study is not correct. Hence the method developed by Noor *et al.* (2011) cannot be compared with the method developed in this study.

In this study, the quartic spline function is used to develop a technique for the solution of the following system:

$$y^{(5)}(x) = \begin{cases} f(x), & a \leq x \leq c, \\ f(x) + y(x)g(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \quad (1)$$

Along with the boundary conditions:

$$\begin{aligned} y(a) = y(b) &= \alpha_0, y^{(1)}(a) = y^{(1)}(b) = \alpha_1, \\ y(c) = y(d) &= \alpha_2, y^{(1)}(c) = y^{(1)}(d) = \alpha_3, \\ y^{(2)}(a) &= \alpha_4, y^{(2)}(c) = y^{(2)}(d) = \alpha_5 \end{aligned} \quad (2)$$

where r and $\alpha_i, i = 0, 1, \dots, 5$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $[c, d]$, respectively. Such type of systems arises in connection with contact, obstacle and unilateral problems.

QUARTIC SPLINE METHOD

To develop the quartic spline approximation S to the problem (1), the interval $[a, b]$ is divided into k equal subintervals (s.t k is divisible by 4 and $k = 4n$), using the grid points $x_i = a + ih; i = 0, 1, \dots, k$ where $h = (b-a)/k$.

Without loss of generality, the knots c and d may be taken as:

$$c = (3a + b)/4, \quad d = (a + 3b)/4$$

The system (1), thus, leads to the following form:

$$y_i^{(5)} = \begin{cases} f_i, & 0 \leq i \leq n-1 \text{ and } 3n+1 \leq i \leq 4n-1, \\ f_i + y(x_i)g_i + r, & n \leq i \leq 3n, \end{cases} \quad (3)$$

where, $f_i = f(x_i)$, $g_i = g(x_i)$, $i = 0, 1, 2, \dots, 4n$. The restriction S_i of S to each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, k - 1$, is defined as:

$$S_i(x) = a_i(x - x_i)^4 + b_i(x - x_i)^3 + c_i(x - x_i)^2 + d_i(x - x_i) + e_i \tag{4}$$

For,

$$\left. \begin{aligned} S_i(x_i) &= y_i, & S_i^{(1)}(x_i) &= m_i, \\ S_i^{(2)}(x_i) &= M_i, & S_i^{(3)}(x_i) &= T_i, \end{aligned} \right\} i = 0, 1, \dots, k \tag{5}$$

And assuming $y(x)$ to be the exact solution of the system (1) and y_i be an approximation to $y(x_i)$, obtained by the spline $S(x_i)$.

After, applying the first and second derivative continuities at the knots, i.e., $S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ for $\mu = 1, 2$, the following consistency relation is derived which is necessary to find the solution of the problem (1):

$$T_{i-2} + 11T_{i-1} + 11T_i + T_{i+1} = \frac{24}{h^3}[-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1}]; \quad i = 2, 3, \dots, k - 1 \tag{6}$$

To make the system (6) consistent with the BVP (1), the finite difference formula of $O(h^4)$ is used, which leads to the following relation:

$$\frac{-T_{i+2} + 16T_{i+1} - 30T_i + 16T_{i-1} - T_{i-2}}{h^2} = 12y_i^{(5)} + O(h^4); \quad i = 2, 3, \dots, k - 2 \tag{7}$$

From Eq. (6) and (7), it can be written as:

$$28T_{i-1} - 8T_i + 28T_{i+1} = \frac{24}{h^3}[-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}] + 12h^2y_i^{(5)} + O(h^6); \quad i = 2, 3, \dots, k - 2 \tag{8}$$

Moreover, following Lucas (1974) for quartic spline, it can be written as:

$$T_{i-1} - 2T_i + T_{i+1} = h^2y_i^{(5)} + O(h^7); \quad i = 2, 3, \dots, k - 1 \tag{9}$$

Equation (8) and (9), leads the following relation:

$$T_i = \frac{1}{2h^3}[-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}] - \frac{1}{3}h^2y_i^{(5)} + O(h^6); \quad i = 2, 3, \dots, k - 2 \tag{10}$$

Substituting T_i in (9) gives the following relation which is used for the quartic spline solution of the system (3):

$$\begin{aligned} 3y_{i-3} - 12y_{i-2} + 15y_{i-1} - 15y_{i+1} + 12y_{i+2} - 3y_{i+3} &= -h^5(2y_{i-1}^{(5)} + 2y_i^{(5)} + 2y_{i+1}^{(5)}), \\ &3 \leq i \leq n - 3, n + 3 \leq i \leq 3n - 3, \\ &3n + 3 \leq i \leq 4n - 3 \end{aligned} \tag{11}$$

The above system does not provide the complete solution of the system (1). Hence, the required equations are obtained as:

$$T_{i-1} = \frac{1}{h^3} \left[\begin{aligned} &-\frac{140}{3}y_{i-1} + 57y_i - 12y_{i+1} + \frac{5}{3}y_{i+2} - 38hy_{i-1}^{(1)} - 12h^2y_{i-1}^{(2)} \\ &+ h^5 \left\{ -\frac{5}{168}y_{i-1}^{(5)} - \frac{227}{420}y_i^{(5)} - \frac{67}{840}y_{i+1}^{(5)} \right\} \end{aligned} \right], \quad i = 1, n + 1, 3n + 1 \tag{12}$$

$$T_i = \frac{1}{h^3} \left[\begin{aligned} &\frac{10933}{3035} y_i - \frac{38859}{6070} y_{i+1} + \frac{10299}{3035} y_{i+2} - \frac{721}{1214} y_{i+3} + \frac{4239}{3035} h y_{i-1}^{(1)} + \frac{1467}{3035} h^2 y_{i-1}^{(2)} \\ &+ h^5 \left\{ \frac{125207}{1274700} y_i^{(5)} + \frac{319169}{1274700} y_{i+1}^{(5)} + \frac{31673}{1274700} y_{i+2}^{(5)} \right\} \end{aligned} \right], i = 1, n+1, 3n+1 \quad (13)$$

$$T_{i+1} = \frac{1}{h^3} \left[\begin{aligned} &-\frac{4229}{1758} y_{i+1} + \frac{4073}{879} y_i - \frac{852}{293} y_{i-1} + \frac{679}{879} y_{i-2} - \frac{163}{1758} y_{i-3} + \frac{224}{293} h y_{i+2}^{(1)} \\ &+ h^5 \left\{ -\frac{5201}{105480} y_{i+1}^{(5)} + \frac{5189}{52740} y_i^{(5)} - \frac{943}{21096} y_{i-1}^{(5)} \right\} \end{aligned} \right], i = n-2, 3n-2, 4n-2 \quad (14)$$

$$T_{i+2} = \frac{1}{h^3} \left[\begin{aligned} &-\frac{23}{2} y_{i+2} + \frac{447}{25} y_{i+1} - \frac{204}{25} y_i + \frac{49}{25} y_{i-1} - \frac{9}{50} y_{i-2} + \frac{168}{25} h y_{i+2}^{(1)} \\ &+ h^5 \left\{ \frac{233}{3000} y_{i+2}^{(5)} + \frac{119}{300} y_{i+1}^{(5)} - \frac{41}{600} y_i^{(5)} \right\} \end{aligned} \right], i = n-2, 3n-2, 4n-2 \quad (15)$$

APPLICATIONS

To illustrate the implementation of the method developed, the following fifth order obstacle boundary value problem can be considered as:

$$\left. \begin{aligned} &-y^{(5)}(x) \geq f(x), \\ &y(x) \geq \psi(x), \\ &(y^{(5)}(x) - f(x))(y(x) - \psi(x)) = 0, \\ &y(-1) = y(1) = y^{(1)}(-1) = y^{(1)}(1) = y^{(2)}(-1) = 0, \quad \text{on } \Omega = [-1, 1], \end{aligned} \right\} \quad (16)$$

where, f is a given force acting on string and $\psi(x)$ is the elastic obstacle. The problem (16) arise in several branches of pure and applied sciences including transportation, equilibrium, optimization, mechanics, structural analysis, fluid flow through porous media and image processing in the medical sciences. Using the ideas and technique of Lewy and Stampacchia (1969), the obstacle problem (16) can be characterized by the following system of variational inequality problem:

$$-y^{(5)} + \mu(y - \psi)(y - \psi) = f(x), \quad -1 < x < 1 \quad (17)$$

$$y(-1) = y(1) = 0, \quad y^{(1)}(-1) = y^{(1)}(1) = y^{(2)}(-1) = \varepsilon \quad (18)$$

where,

ε = A small positive constant

ψ = The obstacle function

$\mu(t)$ = The penalty function defined by:

$$\mu(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (19)$$

Since the obstacle function ψ is known, it is possible to find the exact solution of the problem in the interval $-1/2 \leq x \leq 1/2$.

Assuming that the obstacle function ψ is defined by:

$$\psi(x) = \begin{cases} -1, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ 1, & -1/2 \leq x \leq 1/2. \end{cases} \quad (20)$$

From Eq. (17)-(20), the following system of equations can be obtained as:

$$y^{(5)} = \begin{cases} f, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ -1+y+f, & -1/2 \leq x \leq 1/2, \end{cases} \quad (21)$$

With the boundary conditions:

$$y(-1) = y(-1/2) = y(1/2) = y(1) = 0 \quad (22)$$

$$y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(-1) = y^{(2)}(-1/2) = y^{(2)}(1/2) = \varepsilon \quad (23)$$

And the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $1/2$.

If $\mu(t)$ and $\psi(x)$ are taken as:

$$\mu(t) = \begin{cases} 4, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad (24)$$

$$\psi(x) = \begin{cases} -1/4, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ 1/4, & -1/2 \leq x \leq 1/2, \end{cases} \quad (25)$$

Then the following system of equations can be obtained as:

$$y^{(5)} = \begin{cases} f, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ 1-4y+f, & -1/2 \leq x \leq 1/2. \end{cases} \quad (26)$$

It is to be mentioned that the system of equations associated with the obstacle problem (16) is a special case of the system of fifth order boundary value problem (1):

Numerical examples:

Example 1: The following problem is considered, as:

$$y^{(5)}(x) = \begin{cases} 1, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ 2-4y(x), & -1/2 \leq x \leq 1/2, \end{cases} \quad (27)$$

Together with the boundary conditions at $x = -1, x = 1$ and the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $x = 1/2$. The analytic solution of the system (27) is:

$$y(x) = \begin{cases} 1/480 (1 + 7x + 19x^2 + 25x^3 + 16x^4 + 4x^5), & -1 \leq x \leq -1/2, \\ y(-1) = y(-1/2) = 0, \\ y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(2)}(-1) = 0, \\ 1/2 - \beta_1 \exp(-2^{2/5}x) - \beta_2 \exp(\alpha_1x) \cos(\alpha_2x) \\ - \beta_3 \exp(\alpha_3x) \cos(\alpha_4x) + \beta_4 \exp(\alpha_3x) \sin(\alpha_4x) + \beta_5 \exp(\alpha_1x) \sin(\alpha_2x), & -1/2 \leq x \leq 1/2, \\ y(-1/2) = y(1/2) = 0, \\ y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(2)}(-1/2) = 0, \\ 1/960 (-1 + 8x - 25x^2 + 38x^3 - 28x^4 + 8x^5), & 1/2 \leq x \leq 1, \\ y(1/2) = y(1) = 0, \\ y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(1/2) = 0, \end{cases} \quad (28)$$

where,

$$\begin{aligned} \beta_1 &= 0.0834724026 & 3681168348 & 6236902194 & 4758140005 & 5511002239 & 1306503 \\ \beta_2 &= 0.1972266558 & 7509779339 & 4145018248 & 4356113536 & 1747677047 & 6389696 \\ \beta_3 &= 0.2187804760 & 4580227370 & 6722229445 & 0948297864 & 0274643575 & 3855139 \\ \beta_4 &= 0.0055986289 & 8690219274 & 8619920098 & 9045569432 & 4051110728 & 1886036 \\ \beta_5 &= 0.0316240843 & 4440800135 & 7285362723 & 6256874212 & 2016034882 & 3034071 \\ \alpha_1 &= \left(\frac{1}{22^{3/5}} - \frac{\sqrt{5}}{22^{3/5}} \right), \alpha_2 = \frac{\sqrt{5+\sqrt{5}}}{22^{1/10}}, \alpha_3 = \left(\frac{1}{22^{3/5}} + \frac{\sqrt{5}}{22^{3/5}} \right), \alpha_4 = \frac{\sqrt{5-\sqrt{5}}}{22^{1/10}} \end{aligned}$$

The observed maximum errors (in absolute values) are summarized in Table 1. It is confirmed from the Table 1 that if h is reduced by factor 1/2, then $\|E\|$ is reduced by a factor 1/4, which indicates that the present method gives second order results.

Example 2: The following problem is considered, as:

$$y^{(5)}(x) = \begin{cases} 2, & -1 \leq x \leq -1/2, & 1/2 \leq x \leq 1, \\ -1 + y(x), & -1/2 \leq x \leq 1/2, \end{cases} \quad (29)$$

Together with the boundary conditions at $x = -1, x = 1$ and the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $x = 1/2$.

The analytic solution of the system (29) is:

$$y(x) = \begin{cases} \left. \begin{aligned} &1/240 (1 + 7x + 19x^2 + 25x^3 + 16x^4 + 4x^5), \\ &-1 \leq x \leq -1/2, \\ &y(-1) = y(-1/2) = 0, \\ &y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(2)}(-1) = 0, \end{aligned} \right\} \\ \left. \begin{aligned} &1 - \beta_1 \exp(\alpha_1 x) \cos(\alpha_2 x) - \beta_2 \exp(\alpha_3 x) \cos(\alpha_4 x) \\ &- \beta_3 \exp(\alpha_5 x) + \beta_4 \exp(\alpha_6 x) \sin(\alpha_7 x) + \beta_5 \exp(\alpha_8 x) \sin(\alpha_9 x), \\ &-1/2 \leq x \leq 1/2, \\ &y(-1/2) = y(1/2) = 0, \\ &y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(2)}(-1/2) = 0, \end{aligned} \right\} \\ \left. \begin{aligned} &1/480 (-1 + 8x - 25x^2 + 38x^3 - 28x^4 + 8x^5), \\ &1/2 \leq x \leq 1, \\ &y(1/2) = y(1) = 0, \\ &y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(1/2) = 0, \end{aligned} \right\} \end{cases} \quad (30)$$

where,

$$\begin{aligned} \beta_1 &= 0.4219712533 & 7780818078 & 4844197586 & 4815793042 & 7536008951 & 4777474 \\ \beta_2 &= 0.3639652596 & 0597986678 & 0921476737 & 9425995473 & 4165081679 & 0531421 \\ \beta_3 &= 0.2143239497 & 0950323997 & 7271742174 & 7916155187 & 6011001346 & 5674423 \\ \beta_4 &= 0.0313189768 & 2144723531 & 1388883995 & 0388358841 & 3718494727 & 8662482 \\ \beta_5 &= 0.0329490184 & 3012405416 & 3299461434 & 2663130810 & 7738369730 & 6730612 \\ \alpha_1 &= \left(\frac{-1}{4} + \frac{\sqrt{5}}{4} \right), \alpha_2 = \frac{1}{2} \sqrt{\frac{1}{2}(5 + \sqrt{5})}, \alpha_3 = \left(\frac{-1}{4} - \frac{\sqrt{5}}{4} \right), \alpha_4 = \frac{1}{2} \sqrt{\frac{1}{2}(5 - \sqrt{5})} \end{aligned}$$

Table 1: Maximum absolute errors for problem (27)

h	$ y(x_i) - y_j $
1/14	2.10×10^{-4}
1/28	2.93×10^{-5}
1/56	3.85×10^{-6}
1/112	4.93×10^{-7}
1/224	6.23×10^{-8}

Table 2: Maximum absolute errors for problem (29)

h	The method presented
1/14	4.39×10^{-4}
1/28	6.12×10^{-5}
1/56	8.05×10^{-6}
1/112	1.03×10^{-6}
1/224	1.30×10^{-7}

To analyze the usefulness of the method, the results are summarized in Table 2 and it is evident from the Table 2 that $\|E\|$ is reduced by a factor 1/4, if h is reduced by factor 1/2, which shows that the method is of second order.

CONCLUSION

Quartic spline method is developed for the approximate solutions of system of fifth order BVPs. The obtained numerical results are compared with the exact solution. Noor *et al.* (2011) developed variation of parameters method for the solution of fifth order obstacle problem but the exact solution of the problem given in the study is not correct. Hence the method developed by Noor *et al.* (2011) cannot be compared with the quartic spline method. We showed that the quartic spline method is a powerful mathematical tool for the solution of system of fifth order BVPs. Numerical examples also illustrate the accuracy of the method.

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