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Research Article

Quartic Non-polynomial Spline Solution of a Third Order Singularly Perturbed Boundary Value Problem

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Abstract: In this study, the non-polynomial spline function is used to find the numerical solution of the third order singularly perturbed boundary value problems. The convergence analysis is discussed and the method is shown to have second order convergence. The order of convergence is improved up to fourth order using the improved end conditions. Numerical results are given to describe the efficiency of the method and compared with the method developed by Akram (2012), which shows that the present method is better.

Keywords: Boundary layers, monotone matrices, quartic non-polynomial spline, singularly perturbed boundary value problems, uniform convergence

INTRODUCTION

The purpose of the study is to develop a new spline method for the solution of third order singularly perturbed boundary value problem. The method depends on a non-polynomial spline function which has a trigonometric part and a polynomial part. The following third order self adjoint singularly perturbed boundary value problem is considered, as:

$$\left.\begin{aligned} &L\big(y(x)\big) = -\varepsilon\, y^{(3)}(x) + p(x)y(x) = f(x), \quad p(x)\geq 0 \\ &y(0)=\alpha_0, \;\; y(1)=\alpha_1, \;\; y^{(1)}(0)=\alpha_2 \end{aligned}\right\} \tag{1}$$

or,

$$\left.\begin{aligned} &L\big(y(x)\big) = -\varepsilon\, y^{(3)}(x) + p(x)y(x) = f(x), \quad p(x)\geq 0 \\ &y(0)=\alpha_0, \;\; y(1)=\alpha_1, \;\; y^{(2)}(0)=\alpha_3 \end{aligned}\right\} \tag{2}$$

where, α_0, α_1, α_2 and α_3 are constants and ε is a small positive parameter (0<ε<<1), also $f(x)$ and $p(x)$ are smooth functions and $p(x) = p =$ constant. The spline function has the form $T_n = \{1, x, x^2, cos(kx), (kx), sin(kx)\}$ It is to be noted that k can be real or imaginary. The theory of singularly perturbed problems frequently occur in many branches of engineering and applied sciences, for instance in geophysics, fluid dynamics, Newtonian fluid mechanics, quantum mechanics, gas dynamics, chemical reactions, optimal control theory etc. The numerical treatment of singularly perturbation problems yields major computational difficulties and the usual numerical methods fail to produce accurate results for all independent values of x when ε is very small, owing to the multi-scale character of the solution of singularly perturbation problems. That is there are thin transition layers, where the solution varies rapidly, while away from the layers the solution behaves frequently and varies slowly. Three principle approaches are frequently used to solve such kind of problems numerically, namely the finite difference methods, the finite element methods and spline approximation methods. In this study the third one, namely, the spline approximation method is used.

Howers (1976), Kelevedjiev (2002) and Roos *et al*. (1996) discussed the existence and uniqueness of singularly perturbed Boundary Value Problems (BVPs). Lie (2008) constructed a computational method for singularly perturbed two point BVP in the form of series in reproducing Kernel space. Rashidinia and Mahmoodi (2007) developed the classes of methods for the numerical solution of singularly perturbed two point BVP using non polynomial cubic spline and the method is second order as well as fourth order accurate. Khan *et al*. (2006) used sextic spline to solve second order singularly perturbed BVP and the method is fifth order accurate. Yao and Cui (2007) developed a new algorithm for a class of singularly BVPs in the reproducing Kernel space. Akram (2012) presented a quartic spline solution of a third order singularly perturbed BVP and the method is second order accurate. Akram and Mehak (2012) proposed a quintic spline technique to solve fourth order singularly perturbed BVP.

Consistency relations: To develop the consistency relations the following fourth degree non polynomial spline is considered:

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$$Q_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + \\ + c_i(x - x_i)^2 + d_i(x - x_i) + e_i \tag{3}$$

Defined on $[a, b]$, where $x \epsilon [x_i, x_{i+1}]$ with equally spaced knots, $x_i = a + ih$, $i = 0, 1, \ldots, n$ and $h = \frac{b-a}{n}$. Using the following notations:

$$\left.\begin{array}{ll} Q_i(x_i) = y_i, & Q_i(x_{i+1}) = y_{i+1}, \\ Q_i^{(3)}(x_i) = T_i, & Q_i^{(3)}(x_{i+1}) = T_{i+1}, \\ Q_i^{(1)}(x_i) = N_i. & \end{array}\right\} \tag{4}$$

The coefficients in (3) are determined as:

$$a_i = h^3\left(\frac{T_{i+1} - T_i\cos\theta}{\theta^3 \sin\theta}\right),$$
$$b_i = \frac{-h^3 T_i}{\theta^3},$$
$$c_i = \frac{y_{i+1} - y_i}{h^2} - \frac{h(\cos\theta - 1)(T_{i+1} + T_i)}{\theta^3 \sin\theta} - \frac{N_i\theta^2 + h^2 T_i}{h\theta^2},$$
$$d_i = \frac{N_i\theta^2 + h^2 T_i}{\theta^2},$$
$$e_i = y_i - \frac{h^3(T_{i+1} - T_i\cos\theta)}{\theta^3 \sin\theta}.$$

Applying the first and second derivative continuity at knots, that is, $Q^{(\mu)}{}_{i-1}(x_i) = Q^{(\mu)}{}_i(x_i)$, for $\mu = 1, 2$, the following relations are derived:

$$\frac{\theta^2(N_i + N_{i-1})}{h^2} + (T_{i-1} + T_i) + \frac{2\theta^2(y_{i-1} - y_i)}{h^3} + \frac{2(T_i + T_{i-1})(\cos\theta - 1)}{\theta\sin\theta} = 0,$$
$$2(N_{i-1} - N_i) + \frac{2h^2(T_{i-1} - T_i)}{\theta^2} + \frac{h^2(-T_{i-1} + 2T_i\cos\theta - T_{i+1})}{\theta\sin\theta} + \frac{2(y_{i+1} - 2y_i + y_{i-1})}{h} + \frac{2h^2(\cos\theta - 1)(T_{i-1} - T_{i+1})}{\theta^3\sin\theta} = 0,$$

which leads the following consistency relation in terms of T_i and y_i:

$$\left.\begin{array}{l} -y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} \\ = h^3 T_{i-2}\left(\frac{\cos\theta - 1}{\theta^3\sin\theta} + \frac{1}{2\theta\sin\theta}\right) + h^3 T_{i-1}\left(\frac{1 - \cos\theta}{\theta^3\sin\theta} + \frac{1 - 2\cos\theta}{2\theta\sin\theta}\right) \\ + h^3 T_i\left(\frac{1 - \cos\theta}{\theta^3\sin\theta} + \frac{1 - 2\cos\theta}{2\theta\sin\theta}\right) + h^3 T_{i+1}\left(\frac{\cos\theta - 1}{\theta^3\sin\theta} + \frac{1}{2\theta\sin\theta}\right); \\ i = 2, 3, \ldots, n-1. \end{array}\right\} \tag{5}$$

Equation (5) can also be written, as:

$$y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2} = \alpha h^3 T_{i+1} + \beta h^3 T_i + \beta h^3 T_{i-1} + \alpha h^3 T_{i-2} \quad i = 2, 3, \ldots, n-1. \tag{6}$$

where,

$$\alpha = \left(\frac{\cos\theta - 1}{\theta^3 \sin\theta} + \frac{1}{2\theta\sin\theta}\right)$$

and,

$$\beta = \left(\frac{1 - \cos\theta}{\theta^3\sin\theta} + \frac{1 - 2\cos\theta}{2\theta\sin\theta}\right).$$

It is to be noted that, if $\alpha = 0$ and $\beta = \frac{1}{2}$ then the truncation error of the above equations is $-\frac{1}{240} h^7 y_i^7$. Using Eq. (1) and (6) can be rewritten as:

$$\left.\begin{array}{l} (-\varepsilon + \alpha ph^3)y_{i+1} + (3\varepsilon + \beta ph^3)y_i - (3\varepsilon - \beta ph^3)y_{i-1} + (\varepsilon + \alpha ph^3)y_{i-2} \\ = h^3(\alpha f_{i+1} + \beta f_i + \beta f_{i-1}\alpha f_{i-2}), \quad i = 2, 3, \ldots, n-1. \end{array}\right\} \tag{7}$$

End conditions: The system Eq. (7) consists of (n-2) equations with (n-1) unknowns, so one more equation is required. Following Akram and Siddiqi (2006), the required end condition can be written as:

$$T_0 + a_1 T_1 + a_2 T_2 + T_3 = \frac{1}{h^3}\left[\sum_{i=0}^{4} b_i y_i + c_0 h y_0^{(1)}\right], \tag{8}$$

where, a_1, a_2, c_0 and b_i, $i = 0, 1, 2, 3$ are arbitrary parameters which are to be calculated using method of undetermined coefficients. The end condition of $O(h^5)$ can be calculated, as:

$$T_0 + T_1 + T_2 + T_3 = \frac{1}{h^3}\left[\frac{2}{3}y_0 + 3y_1 - 6y_2 + \frac{7}{3}y_3 + 2hy^{(1)}{}_0\right] \tag{9}$$

Using Eq. (1), Eq. (9) can be rewritten as:

$$(-3\varepsilon + ph^3)y_1 + (6\varepsilon)y_2 + (-\frac{7}{3}\varepsilon + ph^3)y_3 - h^3(f_0 + f_1 + f_3) + (-\frac{2}{3}\varepsilon - ph^3)\alpha_0 - 2\varepsilon h y_0^{(1)} = 0. \tag{10}$$

Similarly the end condition of $O(h^5)$ for the system (2) can be calculated, as:

$$T_0 + 2T_1 + T_2 + T_3 = \frac{1}{h^3}\left[-\frac{43}{11}y_0 + \frac{135}{11}y_1 - \frac{141}{11}y_2 + \frac{49}{11}y_3 - \frac{6}{11}h^2 y^{(2)}{}_0\right], \tag{11}$$

Using Eq. (2), Eq. (11) can be rewritten as:

$$\left.\begin{array}{l} \left(-\frac{135}{11}\varepsilon + 2ph^3\right)y_1 + \left(\frac{141}{11}\varepsilon + ph^3\right)y_2 + \left(-\frac{49}{11}\varepsilon + ph^3\right)y_3 - h^3(f_0 + 2f_1 + f_2 + f_3) \\ + \left(\frac{43}{11}\varepsilon - ph^3\right)\alpha_0 + \frac{6}{11}\varepsilon h^2 y_0^{(2)} = 0. \end{array}\right\} \tag{12}$$

The order of truncation error of end conditions can be improved. Following Akram and Siddiqi (2006), the improved end conditions of O (h^8) for the system (1) can be determined, as:

$$T_0+\frac{97}{7}T_1+\frac{69}{7}T_2+T_3=\frac{1}{h^3}\left[\frac{99}{28}y_0+\frac{176}{7}y_1-\frac{309}{7}y_2+\frac{96}{7}y_3+\frac{7}{4}y_4+15hy^{(1)}_0\right] \quad (13)$$

Using Eq. (1), Eq. (13) can be rewritten as:

$$\left.\begin{aligned}&\left(-\frac{176}{7}\varepsilon+\frac{97}{7}ph^3\right)y_1+\left(\frac{309}{7}\varepsilon+\frac{69}{7}ph^3\right)y_2+\left(-\frac{96}{7}\varepsilon+ph^3\right)y_3-\left(\frac{7}{4}\varepsilon\right)y_4\\&-h^3\left(f_0+\frac{97}{7}f_1+\frac{69}{7}f_2+f_3\right)+\left(-\frac{99}{28}\varepsilon-ph^3\right)\alpha_0+15\varepsilon hy_0^{(1)}=0.\end{aligned}\right\} \quad (14)$$

Similarly end condition of O (h^8) for the system (2) can be calculated, as:

$$T_0+\frac{265}{31}T_1+\frac{213}{31}T_2+T_3=\frac{1}{h^3}\left[-\frac{607}{62}y_0+\frac{944}{31}y_1-\frac{921}{31}y_2+\frac{224}{31}y_3+\frac{113}{62}y_4-\frac{90}{31}h^2y^{(2)}_0\right] \quad (15)$$

Using Eq. (2), Eq. (15) can be rewritten as:

$$\left.\begin{aligned}&\left(-\frac{944}{31}\varepsilon+\frac{265}{31}ph^3\right)y_1+\left(\frac{921}{31}\varepsilon+\frac{213}{31}ph^3\right)y_2+\left(-\frac{224}{31}\varepsilon+ph^3\right)y_3\\&-\left(\frac{113}{62}\varepsilon\right)y_4-h^3\left(f_0+\frac{265}{31}f_1+\frac{213}{31}f_2+f_3\right)+\left(\frac{607}{62}\varepsilon+ph^3\right)\alpha_0+15\varepsilon h^2y_0^{(2)}=0\end{aligned}\right\} \quad (16)$$

CONVERGENCE OF THE METHOD

The system of Eq. (7) and (14) provides the required quartic non polynomial spline solution of the BVP (1), which can be written in matrix form, as:

$$AY-h^3DF=C, \quad (17)$$

where, $A = (a_{ij})$ is a tetradiagonal matrix of order n-1:

$$A=\begin{bmatrix}\frac{97}{7}ph^3-\frac{176}{7}\varepsilon & \frac{69}{7}ph^3+\frac{309}{7}\varepsilon & ph^3-\frac{96}{7}\varepsilon & -\frac{7}{4}\varepsilon & & \\ \beta ph^3-3\varepsilon & \beta ph^3+3\varepsilon & \alpha ph^3-\varepsilon & & & \\ \alpha ph^3+\varepsilon & \beta ph^3-3\varepsilon & \beta ph^3+3\varepsilon & \alpha ph^3-\varepsilon & & \\ & \ddots & \ddots & \ddots & & \\ & \alpha ph^3+\varepsilon & \beta ph^3-3\varepsilon & \beta ph^3+3\varepsilon & \alpha ph^3-\varepsilon & \\ & & \alpha ph^3+\varepsilon & \beta ph^3-3\varepsilon & \beta ph^3+3\varepsilon & \end{bmatrix},$$

$$D=\begin{bmatrix}\frac{97}{7} & \frac{69}{7} & 1 & & \\ \beta & \beta & \alpha & & \\ \alpha & \beta & \beta & \alpha & \\ & \ddots & \ddots & \ddots & \\ & \alpha & \beta & \beta & \alpha \\ & & \alpha & \beta & \beta\end{bmatrix},$$

$C = (c_1, c_2, \ldots, c_{N-2}, c_{N-1})^T$,

$Y = (y_1, y_2, \ldots, y_{N-2}, y_{N-1})^T$

and

$F = (f_1, f_2, \ldots, f_{N-2}, f_{N-1})^T$.

Also,

$$c_1=\left(\frac{99}{28}\varepsilon-ph^3\right)\alpha_0+15\varepsilon hy_0^{(1)}+h^3f_0,$$

$$c_2=\left(-\varepsilon-\alpha ph^3\right)\alpha_0+h^3\alpha f_0,$$

$$c_i=0, \qquad i=3,4,\ldots,n-2,$$

and

$$c_{n-1}=(\varepsilon-\alpha ph^3)\alpha_1+\alpha h^3f_n.$$

If $\overline{Y}=\left[y(x_1),y(x_2),\ldots,y(x_{N-2}),y(x_{N-1})\right]^T$ denotes the exact solution of BVP (1) and Y be the approximate solution then, Eq. (17) can be written, as:

$$A\overline{Y}-h^3DF=T+C, \quad (18)$$

where $T=(t_1,t_2,\ldots,t_{n-1})^T$ with:

$$\left.\begin{aligned}&t_1=\frac{23}{840}\varepsilon h^8y^{(8)}(\xi_1), \quad x_0<(\xi_1)<x_4\\&\text{and}\\&t_i=-\frac{1}{240}\varepsilon h^7y^{(7)}(\xi_i), \quad x_{i-2}<(\xi_i)<x_{i+1}, \quad i=2,3,\ldots,n-1.\end{aligned}\right\} \quad (19)$$

From Eq. (17) and (18), it follows that:

$$A(\overline{Y}-Y)=AE=T, \quad (20)$$

$$E=\overline{Y}-Y=(e_1,e_2,\ldots,e_{N-1})^T. \quad (21)$$

To determine the error bound the row sums S_1, S_2, S_{n-1} of matrix A are calculated, as:

$$\left.\begin{aligned}&S_1=\sum_j a_{1,j}=\frac{99}{7}\varepsilon+\frac{173}{7}ph^3,\\&S_i=\sum_j a_{2,j}=-\varepsilon+(2\beta+\alpha)ph^3,\\&S_{N-2}=\sum_j a_{n-2,j}=(2\beta+2\alpha)ph^3 \qquad i=3,4,\ldots,n-2\\&\text{and}\\&S_{n-1}=\sum_j a_{n-1,j}=\varepsilon+(2\beta+\alpha)ph^3.\end{aligned}\right\} \quad (22)$$

Since the matrix A is observed to be irreducible and monotone, A^{-1} exists and its elements are non negative, therefore from Eq. (20), it can be written as:

$$E=\left(A^{-1}\right)T \quad (23)$$

Also, from the theory of matrices it can be written as:

$$\sum_{i=1}^{N-1}a_{k,i}^{-1}S_i=1 \qquad k=1,2,\ldots,N-1, \quad (24)$$

where $a_{k,i}^{-1}$ is the $(k, i)^{th}$ element of the matrix A^{-1}.

From Eq. (22), it follows that:

$$\sum_{i=1}^{n-1} a_{k,i}^{-1} \le \frac{1}{\min\ S_i} = \frac{1}{(h^3 B_{i_0})} \tag{25}$$

where,

$$B_{i_0} = \frac{1}{h^3} \min S_i > 0 \tag{26}$$

For some i_0 between 1 and n-1.
From Eq. (23) it can be written as:

$$e_k = \sum_{i=1}^{n-1} a_{k,i}^{-1}\ T_i\ , \qquad k = 1, 2, \ldots, n-1. \tag{27}$$

Using Eq. (19) in (27), the following result is obtained:

$$|e_k| \le lh^4 / B_{i_0}, \qquad k = 1, 2, \ldots, n-1, \tag{28}$$

where, l is a constant independent of h. From Eq. (28), it follows that:

$$\|E\| = O(h^4).$$

Similarly, it can be proved that $\|E\| = O(h^4)$ for the solution of BVP (2) given by Eq. (7) and (16). These results are summarized in the following theorems.

Theorem 1: The method given by Eq. (7) and (14) for solving the boundary value problem (1) for sufficiently small h gives a fourth order convergent solution.

Theorem 2: The method given by Eq. (7) and (16) for solving the boundary value problem (2) for sufficiently small h gives a fourth order convergent solution.

On the same fashion, truncation errors of the Eq. (10) and (7) are calculated as:

$$\left.\begin{array}{l} t_1 = -\dfrac{37}{20}\ \varepsilon\, h^5 y^{(5)}(\xi_1), \quad x_0 < (\xi_1) < x_4 \\ \text{and} \\ t_i = -\dfrac{1}{240}\ \varepsilon\, h^7 y^{(7)}(\xi_i), \quad x_{i-2} < (\xi_i) < x_{i+1}, \quad i = 2, 3, \ldots, n-1. \end{array}\right\} \tag{29}$$

Using Eq. (29) in (27), the following result is obtained:

$$|e_k| \le lh^2 / B_{i_0}, \qquad k = 1, 2, \ldots, n-1, \tag{30}$$

where, l is a constant independent of h From Eq. (30), it follows that:

$$\|E\| = O(h^2). \tag{31}$$

Similarly, it can be proved that $\|E\| = O(h^2)$ for the solution of BVP (2) given by Eq. (7) and (12). These results are summarized in the following theorems.

Theorem 3: The method given by Eq. (7) and (10) for solving the boundary value problem (1) for sufficiently small h gives a second order convergent solution.

Theorem 4: The method given by Eq. (7) and (12) for solving the boundary value problem (2) for sufficiently small h gives a second order convergent solution.

To illustrate the implementation of the method and error analysis of the BVP (1) and (2), two examples are discussed in the following section.

NUMERICAL EXAMPLES

Example 1: Consider the following boundary value problem:

$$\left.\begin{array}{l} -\varepsilon\, y^{(3)}(x) + y(x) = 81\varepsilon^2 \cos(3x) + 3\varepsilon \sin(3x), \\ y(0) = 0,\ y^{(1)}(0) = 9\varepsilon,\ y(1) = 3\varepsilon \sin(3), \end{array}\right\} \tag{32}$$

and,

$$\left.\begin{array}{l} -\varepsilon\, y^{(3)}(x) + y(x) = 81\varepsilon^2 \cos(3x) + 3\varepsilon \sin(3x), \\ y(0) = 0,\ y^{(2)}(0) = 9\varepsilon,\ y(1) = 3\varepsilon \sin(3), \end{array}\right\} \tag{33}$$

The analytical solution of system (32) and (33) is:

$$y(x) = 3\varepsilon \sin(3x)$$

The observed maximum errors (in absolute values) associated with y_i, for the problem (32) and (33) corresponding to different values of $\varepsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ are summarized in Table 1 and 2, respectively. Similarly, the observed maximum errors (in absolute values) associated with y_i, for the problem (32) and (33) corresponding to improved end conditions are summarized in Table 3 and 4, respectively.

Remark: The comparison of Table 1 and 3 with 5 shows that the errors in absolute are better than that developed by Akram (2012), corresponding to the problem (32). Similarly, the comparison of Table 2 and 4 with 6 shows that the errors in absolute are better than that developed by Akram (2012), corresponding to the problem (33).

Example 2:

$$\left.\begin{array}{l} -\varepsilon\, y^{(3)} + y(x) = f(x), \\ y(0) = 0,\ y^{(1)}(0) = 0,\ y(1) = 0. \end{array}\right\} \tag{34}$$

where,

Table 1: Maximum absolute errors for problem (32) in y_i

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	7.39×10^{-4}	5.09×10^{-5}	3.26×10^{-6}
$\frac{1}{32}$	3.16×10^{-4}	2.16×10^{-5}	1.39×10^{-6}
$\frac{1}{64}$	1.30×10^{-4}	8.77×10^{-6}	5.60×10^{-7}

Table 2: Maximum absolute errors for problem (33) in y_i

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	1.02×10^{-2}	1.40×10^{-3}	1.73×10^{-4}
$\frac{1}{32}$	3.80×10^{-3}	4.84×10^{-4}	6.15×10^{-5}
$\frac{1}{64}$	1.40×10^{-3}	1.00×10^{-4}	2.00×10^{-5}

Table 3: Maximum absolute errors for problem (32) in y_i corresponding to improved end condition

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	5.70×10^{-7}	5.97×10^{-8}	4.14×10^{-9}
$\frac{1}{32}$	2.49×10^{-7}	2.52×10^{-8}	1.75×10^{-9}
$\frac{1}{64}$	1.00×10^{-7}	9.90×10^{-9}	6.81×10^{-10}

Table 4: Maximum absolute errors for problem (33) in y_i corresponding to improved end condition

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	3.54×10^{-6}	2.94×10^{-8}	6.57×10^{-9}
$\frac{1}{32}$	1.28×10^{-6}	1.27×10^{-8}	2.41×10^{-9}
$\frac{1}{64}$	4.25×10^{-7}	5.33×10^{-9}	8.04×10^{-10}

Table 5: Maximum absolute errors for problem (32) in y_i developed by Akram (2012)

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	6.9×10^{-5}	3.1×10^{-5}	5.4×10^{-6}
$\frac{1}{32}$	3.1×10^{-5}	1.8×10^{-5}	2.8×10^{-6}
$\frac{1}{64}$	4.9×10^{-5}	9.9×10^{-6}	1.4×10^{-7}

Table 6: Maximum absolute errors for problem (33) in y_i developed by Akram (2012)

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	2.5×10^{-3}	1.9×10^{-4}	1.4×10^{-5}
$\frac{1}{32}$	6.8×10^{-4}	5.7×10^{-5}	5.0×10^{-6}
$\frac{1}{64}$	1.2×10^{-4}	1.3×10^{-5}	1.6×10^{-6}

Table 7: Maximum absolute errors for problem (34) in y_i corresponding to improved end condition

ε	N = 10	N = 20	N = 40
$\frac{1}{16}$	2.18×10^{-12}	3.65×10^{-15}	4.62×10^{-18}
$\frac{1}{32}$	1.09×10^{-12}	1.83×10^{-15}	2.31×10^{-18}
$\frac{1}{64}$	5.45×10^{-13}	9.13×10^{-16}	1.16×10^{-18}

$$f(x)=(-1+x)^7x^6(\varepsilon^2(-1+x)x(-216+x(972+(-1026+\varepsilon^2(-1+x)^2)x(1)))\cos(\varepsilon x)+ ((-1+x)^3x^3+3\varepsilon^3(-1+x)^2x^2)(-9+19x)+18\varepsilon(28-17x(12+x(-27+19x))))\sin(\varepsilon x)).$$

The analytical solution of the above problem is:

$$y(x)=(1-x)^{10}(x)^9\sin(\varepsilon x).$$

The observed maximum errors (in absolute values) associated with y_i, for the problem (34) corresponding to different values of $\varepsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ are summarized in Table 7.

CONCLUSION

Quartic non polynomial spline function is used to develop a numerical method for solving third order singularly perturbed boundary value problem. The method is second order convergent and fourth order convergent as well using the improved end conditions. The numerical illustration shows that the developed method maintains a very remarkable high accuracy that makes it very encouraging for dealing with the solution of singularly perturbed boundary value problems.

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