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Research Article

A Generalized Extension of the Hadamard-type Inequality for a Convex Function Defined on the Minimum Modulus of Integral Functions

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Abstract: In this study we extend the Hadamard's type inequalities for convex functions defined on the minimum modulus of integral functions in complex field. Firstly, using the Principal of minimum modulus theorem we derive that m (r) is continuous and decreasing function in R^+. Secondly, we introduce a function t (r) and derived that t (r) and ln$t$ (r) are continuous and convex in R^+. Finally we derive two inequalities analogous to well known Hadamard's inequality by using elementary analysis.

Keywords: Analytic function, Hermite-Hadamard integral inequality, integral function, principal of maximum and minimum modulus

INTRODUCTION

Let $f: I \subseteq R \to R$ is a convex mapping defined on the interval $I \in R$. If a, $b \in I$ and $a<b$, then the following double inequality:

$$f(\frac{a+b}{2}) \le \frac{1}{b-1}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \quad (1)$$

holds. This is called the Hermite-Hadamard inequality. Since its innovation in 1893, Hadamard's inequality (Hadamard, 1893) has been proved to be one of the most practical inequalities in mathematical analysis. A number of papers have been written on this inequality as long as innovative proofs, significant extensions, simplification and plentiful applications (Hadamard, 1893; Heing and Maligranda, 1991/92; Pachpatte, 2003; Mitrinovic, 1970; Tunc, 2012; Dragomir, 1990a, b) and reference cited therein. Hadamard's inequalities deal with a convex function f (x) on [a, b] $\in$ R is between the values of f at the midpoint x = (a+b)/2 and the average of the values of f at the endpoints a and b (Chen, 2012). Fractional integral inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. In the last few decades, much significant development in the classical and new inequalities, particularly in analysis, has been witnessed. These inequalities have many applications in numerical quadrature, transform theory, probability and in statistical problems. The main principle of this study is to establish some integral inequality involving the modulus of complex integral functions. Mainly I derived two integral inequalities for two convex functions both of them defined on minimum modulus for non-zero integral function in Complex field. Throughout this note, we write C, R, R^+ for set of complex numbers, set of real numbers and set of non-negative real numbers, respectively. Beneath we provide some necessary definitions, lemmas and theorem which are closely related to our main result.

Definition 1: If the derivative $f'(z)$ exists at all points z of a region $\Re$, then $f(z)$ is said to be analytic in $\Re$ and referred to as an analytic function in $\Re$ or a function analytic in $\Re$. The terms regular and holomorphic are sometimes used as synonyms for analytic.

Definition 2: If AB and BC are two rectifiable arc of lengths l and l', respectively, which have only the point B in common, the arc AC is evidently also rectifiable, its length being $l + l'$. From this it follows that a Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of the lengths of the regular arcs forming it. Such an arc we call a contour. Also a closed contour means a simple closed Jordan curve which consists of a finite number of regular arcs. Obviously a closed contour is rectifiable.

Definition 3: The maximum and minimum modulus of an integral function usually denoted by M (r) and m (r) respectively and defined by:

$$M(r) = \max_{z\in D}|f(z)| \text{ and } m(r) = \min_{z\in D}|f(z)|$$

where, D is a region bounded by a closed contour C.

Definition 4: An integral function is a function which is analytic for all finite values of z. For example e^z, cos z, sin z and all polynomials are integral functions.

Definition 5: A function $f(x)$ is said to be convex on the closed interval $I \subset R$ if and only if $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$, for all $x, y \in I$ and $0 \leq \lambda \leq 1$ (Copson, 1935).

Lemma 1: If f and g are convex functions and g is non-decreasing then $h(x) = g(f(x))$ is convex. As an example, if $f(x)$ is convex, then so is $e^{f(x)}$ because e^x is convex and monotonically increasing (Copson, 1935; Titchmarsh, 1939).

Theorem 1: (The Principal of Maximum Modulus Theorem) Let $f(z)$ is analytic function, regular in a region D and on its boundary C, where C is a simple closed contour. Then $|f(z)|$ is continuous in D, since:

$$\left\| f(z+h) \right| - \left| f(z) \right\| \leq \left| f(z+h) - f(z) \right|$$

and $|f(z+h) - f(z)| \to 0$ as $h \to 0$. Hence $|f(z)|$ has a maximum value, which is obtained at one or more points. In fact $|f(z)|$ reaches its maximum on the boundary C and not at any interior point of D. We may claim that if $|f(z)| \leq M$ on C, then the same inequality holds at all points of D.

A more precise form of the theorem is as follows.

"If $|f(z)|$ be an analytic function, regular within and on the closed contour C. Let M be the upper bound of $|f(z)|$ on C. Then the inequality $|f(z)| \leq M$ holds everywhere within C. Moreover, $|f(z)| = M$ at a point within C if and only if $|f(z)|$ is constant".

Lemma 2: If $f(z)$ is an integral function and $M(r)$ denotes the maximum value of $|f(z)|$, on the region D: $|z| \leq R$, then $M(r)$ is a steadily increasing continuous function of r (Islam and Rezaul, 2013).

Lemma 3: If $f(z)$ is an integral function and $M(r)$ denotes the maximum modulus of $|f(z)|$, on the region D: $|z| \leq r$, then $M(r)$ is a convex function for any non-negative real values of r (Islam and Rezaul, 2013).

Lemma 4: If $f(z)$ is a non-constant integral function, $|f(0)| \neq 0$, defined on any finite region of the z-plane, and $M(r)$ denotes the maximum modulus of $|f(z)|$ on the region D: $|z| \leq r$. Then $\ln M(r)$ is a convex function of $\ln r$ for any positive real values of r (Islam and Rezaul, 2013).

Theorem 2: If $f(z)$ and $g(z)$ are continuous at $z = z_0$, so also are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)\, g(z)$ and $f(z)/g(z)$, the last only if $g(z_0) \neq 0$. Similar results hold for continuity in a region (Copson, 1935; Titchmarsh, 1939).

MATERIALS AND METHODS

Theorem 3: The most general integral function with no zero is of the form $e^{g(z)}$, where $g(z)$ is an integral function.

Proof: First we assume that $f(z)$ is an integral function and for any values of z, $f(z) \neq 0$. Suppose that $F(z)$ is an integral function and defined by:

$$F(z) = \frac{f'(z)}{f(z)} \tag{2}$$

We may choose z_0 ($\neq 0$) and suppose that $\alpha = \arg\{f(z_0)\}$, $\mu = \ln|f(z_0)| + i\alpha$.

Therefore:

$$e^{\mu} = e^{\ln|f(z_0)| + i\alpha} = |f(z_0)| e^{i \arg\{f(z_0)\}} = f(z_0)$$

Now we consider the integral $g(z)$ and defined by:

$$g(z) = \mu + \int_{z_0}^{z} \frac{f'(w)}{f(w)} dw \tag{3}$$

Then $g(z)$ is analytic and we obtain that $g'(z) = f'(z) / f(z)$ and $g(z_0) = \mu$.

Let us consider:

$$h(z) = \frac{e^{g(z)}}{f(z)} \tag{4}$$

The function is $h(z)$ is analytic because $f(z)$, $g(z)$ are analytic.

We obtain:

$$h'(z) = \frac{g'(z)e^{g(z)}}{f(z)} - \frac{f'(z)e^{g(z)}}{\{f(z)\}^2}$$
$$= \frac{f'(z)e^{g(z)}}{\{f(z)\}^2} - \frac{f'(z)e^{g(z)}}{\{f(z)\}^2} = 0$$

Therefore we state that $h(z)$ is a constant function.

Let us consider:

$$h(z) = M$$
$$\Rightarrow \frac{e^{g(z)}}{f(z)} = M \tag{5}$$

In (5) we may substitute a particular value for z, say $z = z_0$. We obtain:

$$M = \frac{e^{g(z_0)}}{f(z_0)} = \frac{e^{\mu}}{e^{\mu}} = 1$$

From (5), we get:

$$f(z) = e^{g(z)}$$

This completes the proof.

Theorem 4: (The Principal of Minimum Modulus Theorem) If $f(z)$ is a non-constant integral function without zeros within the region bounded by a closed contour C, then $|f(z)|$ obtained its minimum value at a point on the boundary of C, i.e., if m is the minimum value of $|f(z)|$ on C, then the inequality $|f(z)| \geq m$ holds for any z lies inside C.

Proof: First we assume that $f(z)$ is a non-constant integral function without zeros within the region bounded by a closed contour C, i.e., $f(z) \neq 0$. Consequently we show that $1/f(z)$ represent a non-constant analytic function within the region bounded by a closed contour C. By the principal of maximum modulus theorem we say that the maximum value of $1/|f(z)|$ must attain on the boundary of C. Hence the minimum value of $|f(z)|$ also attain on the boundary of C. Otherwise which contradict the principal of maximum modulus theorem, since $1/f(z)$ is a non-constant analytic function. So we can say that, if m refer the minimum value of $|f(z)|$, the inequality holds $|f(z)| \geq m$, for any z lies inside C.
This completes the proof.

Lemma 5: If $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$, on the region D: $|z| \leq r$, then $m(r)$ is decreasing and continuous function in R^+.

Proof: First we choose r_1 and r_2 such that $0 \leq r_1 < r_2 \leq R$. Let $m(r_1)$ and $m(r_2)$ denote the minimum modulus of $|f(z)|$ on the regions D: $|z| \leq r_1$ and D: $|z| \leq r_2$ respectively. It is clear that $D_1 \subset D_2$. By the principal of minimum modulus theorem we say that $m(r_1)$ and $m(r_2)$ obtained on the boundary of D_1 and D_2. Suppose that $m(r_1)$ attained at z_1 and $m(r_2)$ attained at z_2. Now z_1 lies on the boundary of D_1, hence it is an interior point of region D_2. Using the principal of minimum modulus theorem, we get:

$$m(r_1) = |f(z_1)| > m(r_2) \tag{6}$$

Therefore, $m(r)$ is a decreasing function of r, since r_1 and r_2 are arbitrary.

To complete the proof we need to show that $m(r)$ is a continuous function of r, i.e., we need show that for any δ (>0) there exits $\in > 0$ such that $|m(r_1) - m(r_2)| < \in$, whenever $|r_1 - r_2| < \delta$. Given that $f(z)$ is analytic hence for any z, satisfying the inequality $|z - z_0| < \delta$, we get:

$$\big||f(z)| - |f(z_0)|\big| \leq |f(z) - f(z_0)| < \in \tag{7}$$

This implies $f(z)$ is continuous in R^+. Hence the modulus of integral functions always continuous in R^+.

Let us consider $m(r_1)$ and $m(r_1)$ attained at z_1 and z_2 respectively, then we get $|z_1| = r_1$ and $|z_2| = r_2$. By (7), for $|r_1 - r_2| = ||z_1|-|z_2|| \leq |z_1-z_2| < \delta$, we obtain:

$$|m(r_1) - m(r_2)| = \big||f(z_1)| - |f(z_2)|\big|$$
$$\leq |f(z_1) - f(z_2)| < \in$$

Hence $m(r)$ is continuous function.

Therefore $m(r)$ is decreasing and continuous function of r, $r \in R^+$.
This completes the proof.

Lemma 6: Let $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$, on the region D: $|z| \leq$r. Let t: $R^+ \rightarrow R$ and defined by $t(r) = 1/m(r)$, then $t(r)$ is increasing and continuous function in R^+.

Proof: First we choose r_1 and r_2 such that $0 \leq r_1 < r_2 \leq R$. Let $m(r_1)$ and $m(r_1)$ denote the minimum modulus of $|f(z)|$ on the regions D: $|z| \leq r_1$ and D: $|z| \leq r_2$, respectively. By (6), we get:

$m(r_1) > m(r_2)$
i.e., $1/m(r_1) < 1/m(r_2)$
Therefore $t(r_1) < t(r_2)$

Hence $t(r)$ is increasing function. Also we state that $t(r)$ is continuous function of r. Therefore $t(r)$ is increasing and continuous function in R^+.
This completes the proof.

Theorem 5: Let $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$ on the region D: $|z| \leq r$. If $r_1 < r_2 < r_3$, then we show that $t_i = t(r_i) = 1/m(r_i)$, for i = 1, 2, 3, satisfy the inequality:

$$t_2^{\ln\left(\frac{r_3}{r_1}\right)} \leq t_1^{\ln\left(\frac{r_3}{r_2}\right)} t_3^{\ln\left(\frac{r_2}{r_1}\right)}$$

Proof: First we assume that $F(z) = z^{\alpha} f(z)$ and α is a real constant to be fixed later. The function $F(z)$ is not, in general, single-valued. Then $F(z)$ is regular in the ring-shaped region between $|z| = r_1$ and $|z| = r_3$ and $|F(z)|$ is single-valued.

Consider the region G: $r_1 \leq |z| \leq r_3$. Now we can reduce the doubly connected region in a simple connected region by introducing a cut AB along the negative part of the real axis. In the simply connected region G bounded by the curve $AGFBCDBA$, $F(z)$ is regular. Hence by the principal of minimum modulus theorem we state that $t(r)$ attain on the boundary of G, because the minimum value of $|F(z)|$ never attain at any point on the cut AB excepting A and B, Otherwise, if we change the cut then the minimum value of $|F(z)|$ attain at an interior point, which contradict the principal minimum modulus theorem. Thus we conclude that the minimum value of $|F(z)|$ attain at one or more points on $|z| = r_1$ or $|z| = r_3$.

Therefore $t(r)$ attain on the boundary of G and it is equal to $\max\{r_1^{\alpha} t_1, r_3^{\alpha} t_3\}$. So for any $z = re^{i\theta} \in G$, we get:

$$\mathrm{t(r)} \le \max\{r_1^{\alpha} t_1, r_3^{\alpha} t_3\} \quad (8)$$

Let $r_1<r_2<r_3$. Then the value of $t(r_2)$ on the circle $|z| = r_2$ is $r_2^{\alpha} t_2$. Hence by (8), we get:

$$\mathrm{t(r_2)} = r_2^{\alpha} t_2 \le \max\{r_1^{\alpha} t_1, r_3^{\alpha} t_3\} \quad (9)$$

We fixed the value of α such that:

$$r_1^{\alpha} t_1 = r_3^{\alpha} t_3$$

Implies:

$$\alpha = -\frac{\ln\left(\frac{t_3}{t_1}\right)}{\ln\left(\frac{t_1}{t_3}\right)}$$

Therefore, we can write:

$$r_2^{\alpha} t_2 \le r_1^{\alpha} t_1$$

$$\Rightarrow t_2^{\ln\left(\frac{r_3}{r_1}\right)} \le t_1^{\ln\left(\frac{r_3}{r_1}\right)} \left(\frac{r_2}{r_1}\right)^{\ln\left(\frac{t_3}{t_1}\right)}, \Rightarrow t_2^{\ln\left(\frac{r_3}{r_1}\right)} \le t_1^{\ln\left(\frac{r_3}{r_1}\right)} \left(\frac{t_3}{t_1}\right)^{\ln\left(\frac{r_2}{r_1}\right)}$$

$$\Rightarrow t_2^{\ln\left(\frac{r_3}{r_1}\right)} \le t_1^{\ln\left(\frac{r_3}{r_2}\right)} t_3^{\ln\left(\frac{r_2}{r_1}\right)} \quad (10)$$

This completes the proof.

Lemma 7: If $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$, on the region G: $|z| \le r$, then $t(r) = 1/m(r)$ and $\ln t(r)$ are convex functions of r and $\ln r$ respectably.

Proof: Let $r_1<r_2$, $r_3 \le R$ and $m(r_i)$ be the minimum modulus of $|f(z)|$ on the region bounded by the circles $|z| = r$, for $i = 1, 2, 3$. Let $t_i = t(r_i) = 1/m(r_i)$, by (10) we get:

$$t_2^{\ln\left(\frac{r_3}{r_1}\right)} \le t_1^{\ln\left(\frac{r_3}{r_2}\right)} t_3^{\ln\left(\frac{r_2}{r_1}\right)}$$

The sign of equality will occur only if the function $f(z)$ is constant multiple of a power of z. Excluding this case, we get:

$$t_2^{\ln\left(\frac{r_3}{r_1}\right)} < t_1^{\ln\left(\frac{r_3}{r_2}\right)} t_3^{\ln\left(\frac{r_2}{r_1}\right)}$$

Taking logarithm on both sides, we obtain:

$$\ln\left(\frac{r_3}{r_1}\right)\ln t_2 < \ln\left(\frac{r_3}{r_2}\right)\ln t_1 + \ln\left(\frac{r_2}{r_1}\right)\ln t_3$$

$$\Rightarrow \ln t_2 < \frac{\ln r_3 - \ln r_2}{\ln r_3 - \ln r_1}\ln t_1 + \frac{\ln r_2 - \ln r_1}{\ln r_3 - \ln r_1}\ln t_3 \quad (11)$$

Since $\ln t(r)$ is a continuous function of $\ln r$ and so if we put $x = \ln r$, then we get $\ln t(r) = \varphi(\ln r) = \varphi(x)$. Also consider $x_i = \ln r_i$, for $i = 1, 2, 3$, then:

$$\ln t_i = \ln t_i(r_i) = \varphi(\ln r_i) = \varphi(x_i)$$

So we obtain the following inequality from (11):

$$\varphi(x_2) < \frac{x_3 - x_2}{x_3 - x_1}\varphi(x_1) + \frac{x_2 - x_1}{x_3 - x_1}\varphi(x_3)$$
$$= \lambda\varphi(\mathrm{x}_1) + (1-\lambda)\varphi(x_3).$$

where $\lambda = (x_3 - x_2) / (x_3 - x_1) <1$, since $x_3<x_2<x_1$.

Hence $\varphi(x)$ is a convex function of x i.e., $\ln t(r)$ is a convex function of $\ln r$.

To complete the proof need to show that $t(r)$ is a convex function of r. By lemma 1 we can say $t(r)$ is convex, since $\ln t(r)$ is increasing and convex function and $e^{\ln t(r)} = t(r)$. This completes the proof.

RESULTS AND DISCUSSION

Theorem 6: If $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$, on the region D: $|z| \le r$. Then for $a, b \in I \subset [0, \infty)$, with $a<b$, we derive the following inequality:

$$t\left(\frac{a+b}{2}\right) \le \frac{1}{b-1}\int_a^b t(x)dx \le \frac{t(a)+t(b)}{2}$$

where $t(r) = 1/m(r)$.

Proof: Let $T: I \subset [0, \infty) \rightarrow \mathrm{R}$ is a mapping on the interval I of real numbers, defined by $t(r) = 1/m(r)$. In lemma 7 we proved that $t(r)$ is a convex function in R^+. Now for any $a, b \in I \subset [0, \infty)$, with $a<b$, we get $t(r)$ is a convex function on the interval I. Hence by the Hermite-Hadamard inequality on convex function, we derive the following double inequality:

$$t(\frac{a+b}{2}) \leq \frac{1}{b-1}\int_a^b t(x)dx \leq \frac{t(a)+t(b)}{2} \quad (12)$$

The proof of this theorem is completed.

Theorem 7: If $f(z)$ is a non-constant integral function, without zeros, defined on any finite region of the z-plane and $m(r)$ denotes the minimum value of $|f(z)|$, on the region D: $|z| \leq r$, then for $a, b \in I \subset [0, \infty)$, with $a<b$, we derive the following inequality:

$$\ln t(\frac{a+b}{2}) \leq \frac{1}{b-1}\int_a^b \ln t(x)dx \leq \frac{\ln t(a)+\ln t(b)}{2}$$

where $t(r) = 1/m(r)$.

Proof: Let $T: I \subset [0, \infty) \rightarrow$ R is a mapping on the interval I of real numbers, defined by $T(r) = \ln t(r) = 1/\ln m(r)$. In lemma 7 we proved that $\ln t(r)$ is a convex function in R^+. Now for any $a, b \in I \subset [0, \infty)$ with $a<b$, we get $\ln t(r)$ is a convex function on the interval I. Hence by the Hermite-Hadamard inequality on convex function, we derive the following double inequality:

$$\ln t(\frac{a+b}{2}) \leq \frac{1}{b-1}\int_a^b \ln t(x)dx \leq \frac{\ln t(a)+\ln t(b)}{2} \quad (13)$$

The proof of this theorem is completed.

CONCLUSION

Inequalities (12) and (13) indicated our final results. These are the extension of the Hadamard's type inequalities (1) for the function $t(r)$ and $\ln t(r)$. In the next study we will give the applications on these results by obtaining some Hadamard's-type inequality for Meromorphic functions in complex field.

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