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Research Article

Shape Preserving Interpolation using Rational Cubic Spline

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Abstract: This study proposes new C^1 rational cubic spline interpolant of the form cubic/quadratic with three shape parameters to preserves the geometric properties of the given data sets. Sufficient conditions for the positivity and data constrained modeling of the rational interpolant are derived on one parameter while the remaining two parameters can further be utilized to change and modify the final shape of the curves. The sufficient conditions ensure the existence of positive and constrained rational interpolant. Several numerical results will be presented to test the capability of the proposed rational interpolant scheme. Comparisons with the existing scheme also have been done. From all numerical results, the new rational cubic spline interpolant gives satisfactory results.

Keywords: Continuity, parameters, positivity preserving, rational cubic spline, shape preserving

INTRODUCTION

Shape preserving data interpolation and approximation are important in computer graphics, geometric modeling and scientific visualization. There are many characteristics of the geometric for data sets. For examples, the given data sets might be monotone, convex and positive. In positivity preserving, the proposed interpolant (either rational or polynomial) must be able to produce the interpolating curves and/or surfaces that will preserves the characteristics of the data namely positivity. There are many situation arise in our daily life involving positivity. Notably, the distributions of wind energy and rainfall measurement are always having positive values. Thus, it is important that when the data is to be visualized for computer display, the resultant curves or surfaces must retain its geometric shape properties (in our case it is positive and data constrained above any straight line).

There are many research papers concerning about positivity preservation either by using cubic spline interpolation or rational spline interpolation. One of the early finding in positivity preserving by using cubic spline interpolation can be found in Dougherty *et al.* (1989). The authors give the sufficient conditions for the positivity of cubic and quintic spline polynomial. Meanwhile, Brodlie and Butt (1991) and Butt and Brodlie (1993) have used cubic polynomials for positivity and convexity preserving by inserting one or two extra notes in interval in which the positivity and/or convexity of the curves are found. By having inserting extra knots, the computation will be increased, hence is not suitable to assists the user in controlling the shape of the data sets. Besides the use of cubic or quintic spline polynomial, several researchers have proposed rational spline interpolant to preserves the positivity of the given data sets. For examples, Sarfraz (2002), Sarfraz *et al.* (2001), Hussain and Sarfraz (2008) and Abbas *et al.* (2012a) studied the use of rational cubic interpolant (cubic numerator and cubic denominator) for preserving the positive data. Meanwhile Sarfraz *et al.* (2010) studied positivity preserving for curves and surfaces by utilizing rational cubic spline with quadratic denominator. In Hussain *et al.* (2011), the rational cubic spline with quadratic denominator have been used for positivity and convexity preserving with degree attained is C^2. Hussain and Ali (2006) have discussed the positivity preserving by using rational cubic spline originally proposed by Tian *et al.* (2005). Meanwhile Hussain and Hussain (2006) have extended the results in Hussain and Ali (2006) to preserves the data above the straight line and positive surfaces interpolating problem. Abbas *et al.* (2012b) have proposed new rational cubic spline (cubic/quadratic) with three parameters for monotonicity preserving interpolation and those method has been extend to positivity preserving in Abbas *et al.* (2013). Motivated by the works of Tian *et al.* (2005), Hussain and Ali (2006), Hussain and Hussain (2006) and Abbas *et al.* (2012b), in this study, with the authors will proposed new rational cubic spline with three parameters and the rational cubic spline of Tian *et al.* (2005) is a special case of our rational interpolant. Numerical comparison also has been done with existing shape preserving

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interpolation methods. From numerical results, it can be clearly seen that, our proposed rational cubic spline provides good alternative to the existing rational interpolant scheme.

The main scientific contribution this study is as follows:

- In this study a new rational cubic spline (cubic/quadratic) with three parameters has been used for positivity preserving while in Hussain and Ali (2006), Hussain and Hussain (2006) and Hussain *et al*. (2011), the rational cubic spline (cubic/quadratic) with two parameters haven used for positivity preserving.
- The rational cubic spline reduces to the rational cubic spline of Tian *et al*. (2005) when one the parameter is equal to zero, i.e., $\gamma_i = 0$. Indeed, when $\gamma_i = 0$, our positivity-preserving and data constrained interpolation reduces to the works of Hussain and Ali (2006) and Hussain and Hussain (2006), respectively.
- The degree smoothness attained is this study is C^1 whereas in Abbas *et al*. (2013) the degree smoothness attained is C^2. Furthermore; the work in this study can easily be extended to produce C^2 rational cubic spline interpolant.
- Numerical comparison between the proposed rational cubic schemes with two existing methods also has been done.
- Similar to the works by Hussain and Ali (2006), Hussain and Hussain (2006) and Hussain *et al*. (2011), etc., our method also do not required any extra knots. The cubic spline scheme by Butt and Brodlie (1993) and Brodlie and Butt (1991) requires one or two extra knots to be inserted in the interval in which the positivity of the given data sets is not preserved.
- Our rational schemes is based from spline function while in Bashir and Ali (2013) and Ibraheem *et al*. (2012) the interpolant is based from trigonometric spline. Our method works well and guarantee preserves the positivity of the data.
- Finally, in this study two algorithms have been given for the computer implementation while there are no algorithms for computer implementation in Hussain and Ali (2006) and Hussain and Hussain (2006).

MATERIALS AND METHODS

Rational cubic spline interpolant: This section will introduce a new rational cubic spline interpolant with three parameters. Let us assume that, given the set of data, Suppose $\{(x_i, f_i),\ i = 0, 1,\ldots,n\}$ is a given set of data points, where $x_0<x_1<\ldots<x_n$. Let $h_i = x_{i+1} - x_i$, $\Delta_i = \frac{(f_{i+1}-f_i)}{h_i}$ and a local variable, $\theta = \frac{(x-x_i)}{h_i}$ where $0\le\theta\le1$.

For $x \in [x_i, x_{i+1}], i = 0,1,2,...,n-1$:

$$s(x) \equiv s_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{1}$$

where,

$$P_i(\theta) = A_0(1-\theta)^3 + A_1\theta(1-\theta)^2 + A_2\theta^2(1-\theta) + A_3\theta^3,$$
$$Q_i(\theta) = (1-\theta)^2\alpha_i + \theta(1-\theta)(2\alpha_i\beta_i + \gamma_i) + \theta^2\beta_i.$$

The following conditions are required to ensure that the rational function in (1) has C^1 continuity:

$$s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1},$$
$$s_i^{(1)}(x_i) = d_i, \quad s_i^{(1)}(x_{i+1}) = d_{i+1}, \tag{2}$$

Now, by some algebraic manipulation to the C^1 condition in (2), the unknowns A_i, $i = 0, 1, 2, 3$ are given as follows:

$$A_0 = \alpha_i f_i$$
$$A_1 = (2\alpha_i\beta_i + \alpha_i + \gamma_i)f_i + \alpha_i h_i d_i, A_2 = (2\alpha_i\beta_i + \beta_i + \gamma_i)f_{i+1} - \beta_i h_i d_{i+1},$$
$$A_3 = \beta_i f_{i+1}$$

where, $s^{(1)}_i(x)$ denotes derivative with respect to x and d_i denotes the derivative value which is given at the k not x_i, $i = 0, 1, 2,\ldots, n$. The parameters $\alpha_i, \beta_i>0, \gamma_i\ge0$. The data dependent sufficient conditions on the parameters α_i and β_i will be developed in order to preserves the positivity on the entire interval $[x_i, x_{i+1}]$, $i = 0, 1, 2,\ldots, n$-1.

Some observation to the new rational cubic spline interpolant given in (1) can be described as follows:

- When $\alpha_i>0, \beta_i>0, \gamma_i = 0$ the rational interpolant in (1) reduce to the rational spline of the form cubic/quadratic by Tian *et al*. (2005).
- When $\alpha_i = \beta_i = 1, \gamma_i = 0$ the rational cubic interpolant in (1) is a standard cubic Hermite spline given as follow:

$$S(x) = (1-\theta)^2(1+2\theta)f_i + \theta^2(3-2\theta)f_{i+1} + \theta(1-\theta)^2 d_i - \theta^2(1-\theta)d_{i+1} \tag{3}$$

- When $\alpha_i \to 0, \beta_i \to 0$ rational interpolant in (1) converges to straight line given below:

$$\lim_{\alpha_i,\beta_i\to0} s(x) = (1-\theta)f_i + \theta f_{i+1} \tag{4}$$

Furthermore the rational interpolant in (1) can be rewritten as:

$$s(x) = (1-\theta)f_i + \theta f_{i+1} + \frac{h_i\theta(1-\theta)\left[\alpha_i(d_i-\Delta_i)(1-\theta) + \beta_i(\Delta_i - d_{i+1})\theta\right]}{Q_i(\theta)}. \tag{5}$$

The shape parameters α_i, β_i $i = 0, 1, 2, \ldots, n-1$ are free parameter (independent) while the positivity and monotonicity constrained will be derived from the other parameter γ_i (dependent). The two parameters α_i, β_i can be used to refine and modify the final shape of the interpolating curve. This will be useful for shape control of interpolating curves.

Determination of derivatives: For the scalar data sets, the derivative parameters must be estimated by using mathematics formulation. The original derivations to estimate the first derivative are given by Delbourgo and Gregory (1985) and Sarfraz *et al*. (1997). Among those methods are Arithmetic Mean Method (AMM), Geometric Mean Method (GMM) and Harmonic Mean Method (HMM). For our purpose in this study, the Arithmetic Mean Method (AMM) will be used. This is because, AMM is a simple method and suitable for positive data. It also provides very visual pleasing results as can be seen in Results and Discussion Section later. Below the mathematical formula of AMM.
At the end points x_0 and x_n:

$$d_0 = \Delta_0 + \left(\Delta_0 - \Delta_1\right)\left(\frac{h_0}{h_0 + h_1}\right) \tag{6}$$

$$d_n = \Delta_{n-1} + \left(\Delta_{n-1} - \Delta_{n-2}\right)\left(\frac{h_{n-1}}{h_{n-1} + h_{n-2}}\right) \tag{7}$$

At the interior points, x_i, $i = 1, 2, \ldots, n-1$, the values of d_i are given as:

$$d_i = \frac{h_{i-1}\Delta_i + h_i\Delta_{i-1}}{h_{i-1} + h_i} \tag{8}$$

Positivity-preserving using rational cubic spline interpolant: The proposed rational cubic spline interpolant (cubic/quadratic) with three parameters in below section does not always preserves the positivity of the positive data. The ordinary cubic spline interpolation and cubic Hermite spline also does not guarantee to preserve the positivity of the data sets and it will destroy the characteristics of the data sets. These shape violations can be seen clearly from Fig. 1 and 2, respectively. The user may manipulate the values of the shape parameters α_i, β_i, γ_i, $i = 0, 1, \ldots, n-1$ by trial and error basis in order to preserves the positivity of the data. But this approach is really time consuming and obviously it is not recommended to the user.

Following the same idea by Sarfraz (2002), the automated choice of the shape parameter γ_i will be derived from the other two parameters α_i, β_i and the data dependent sufficient conditions for positivity of the rational interpolant defined by Eq. (1) will be developed.

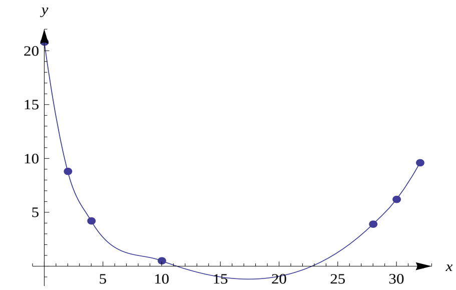

Fig. 1: Default cubic spline polynomial ($\alpha_i = \beta_i = 1$, $\gamma_i = 0$) for data in Table 3

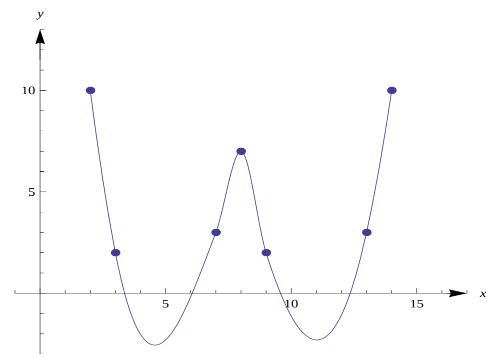

Fig. 2: Default cubic spline polynomial ($\alpha_i = \beta_i = 1$, $\gamma_i = 0$) for data in Table 4

Assuming that the strictly positive set of data (x_i, f_i), $i = 0, 1, \ldots, n$ are given, so that:

$$x_0 < x_1 < \ldots < x_n \tag{9}$$

and,

$$f_i > 0, \; i = 0, 1, \ldots, n \tag{10}$$

Now, the sufficient conditions for positivity of piecewise rational cubic spline (cubic/quadratic) with C^1 continuity will be developed. The main idea is that, in order to preserve the positivity of $s(x)$, the suitable values of shape parameter γ_i in each corresponding interval must be chosen and assigned properly. For all α_i, $\beta_i > 0$ and $\gamma_i \geq 0$ the denominator $Q_i(\theta) > 0$, $i = 0, 1, \ldots, n-1$, therefore the positivity of rational interpolant in (1) solely depends to the positivity of cubic spline polynomial $P_i(\theta)$, $i = 0, 1, \ldots, n-1$. The cubic polynomial $P_i(\theta)$, $i = 0, 1, \ldots, n-1$ can be rewritten as follows:

$$P_i\left(\theta\right) = B_i\theta^3 + C_i\theta^2 + D_i\theta + E_i,$$

where,

$$B_i = \alpha_i h_i d_i + \beta_i h_i d_{i+1} + 2\alpha_i\beta_i f_i + \gamma_i f_i - 2\alpha_i\beta_i f_{i+1} - \gamma_i f_{i+1},$$
$$C_i = -2\alpha_i h_i d_i - \beta_i h_i d_{i+1} + \alpha_i f_i - 4\alpha_i\beta_i f_i - 2\gamma_i f_i + \beta_i f_{i+1} + 2\alpha_i\beta_i f_{i+1} + \gamma_i f_{i+}$$
$$D_i = \alpha_i h_i d_i - 2\alpha_i f_i + 2\alpha_i\beta_i f_i + \gamma_i f,$$
$$E_i = \alpha_i f_i.$$

For strictly monotone data sets given in (10), the following theorem gives the sufficient conditions for positivity preserving.

Theorem 1: For a strictly positive data defined in (10), the rational cubic interpolant defined over the interval $[x_0, x_n]$ is positive if in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1,\ldots, n-1$ the following sufficient conditions are satisfied:

$$\gamma_i > \text{Max}\left\{0, -\alpha_i\left[\frac{h_i d_i + (2\beta_i + 1) f_i}{f_i}\right], \beta_i\left[\frac{h_i d_{i+1} - (2\alpha_i + 1) f_{i+1}}{f_{i+1}}\right]\right\} \tag{11}$$

Proof: To prove Theorem 1, the following Proposition from Schmidt and Hess (1988) is required.

Proposition 1: (Positivity of Cubic polynomial).

For the strict inequality positive data in (10), $P_i(\theta) > 0$ if and only if:

$$\left(P_i'(0), P_i'(1)\right) \in R_1 \cup R_2 \tag{12}$$

where,

$$R_1 = \left\{(a,b) : a > \frac{-3P_i(0)}{h_i}, b < \frac{3P_i(1)}{h_i}\right\}, \tag{13}$$

$$R_2 = \left\{\begin{array}{l}(a,b) : 36 f_i f_{i+1}\left(a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2\right) \\ \quad + 3\left(f_{i+1}a - f_i b\right)\left(2h_i ab - 3f_{i+1}a + 3f_i b\right) \\ \quad + 4h_i\left(f_{i+1}a^3 - f_i b^3\right) - h_i^2 a^2 b^2 > 0\end{array}\right\} \tag{14}$$

where, $a = P'_i(0)$, $b = P'_i(1)$ and $P_i(0) = \alpha_i f_i$, $P_i(1) = \beta_i f_{i+1}$. Now, by differentiating $P_i(\theta)$ from (1) with respect to θ, the results are:

$$P_i'(0) = \frac{-3\alpha_i f_i + f_i(2\alpha_i\beta_i + \alpha_i + \gamma_i) + \alpha_i h_i d_i}{h_i}$$

and,

$$P_i'(1) = \frac{-f_{i+1}(2\alpha_i\beta_i + \beta_i + \gamma_i) + \beta_i h_i d_{i+1} + 3\beta_i f_{i+1}}{h_i}$$

Now from Proposition 1, it can be deduced that $(P'_i(0), P'_i(1)) \in R_1$ if:

$$P_i'(0) = \frac{-3\alpha_i f_i + f_i(2\alpha_i\beta_i + \alpha_i + \gamma_i) + \alpha_i h_i d_i}{h_i} > \frac{-3\alpha_i f_i}{h_i} \tag{15}$$

and,

$$P_i'(1) = \frac{-f_{i+1}(2\alpha_i\beta_i + \beta_i + \gamma_i) + \beta_i h_i d_{i+1} + 3\beta_i f_{i+1}}{h_i} < \frac{3\beta_i f_{i+1}}{h_i} \tag{16}$$

The inequality (15) and (16) leads to the following relations:

$$\gamma_i > -\alpha_i\left[\frac{h_i d_i + (2\beta_i + 1) f_i}{f_i}\right] \tag{17}$$

and,

$$\gamma_i > \beta_i\left[\frac{h_i d_{i+1} - (2\alpha_i + 1) f_{i+1}}{f_{i+1}}\right] \tag{18}$$

Now, Eq. (17) and (18) can be combined to the following sufficient conditions:

$$\gamma_i > \text{Max}\left\{0, -\alpha_i\left[\frac{h_i d_i + (2\beta_i + 1) f_i}{f_i}\right], \beta_i\left[\frac{h_i d_{i+1} - (2\alpha_i + 1) f_{i+1}}{f_{i+1}}\right]\right\}$$

This complete the proof of Theorem 1.
Furthermore, $(P'_i(0), P'_i(1)) \in R_2$ if:

$$\phi(a,b) = \left\{\begin{array}{l}(a,b) : 36 f_i f_{i+1}\left(a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2\right) \\ \quad + 3\left(f_{i+1}a - f_i b\right)\left(2h_i ab - 3f_{i+1}a + 3f_i b\right) \\ \quad + 4h_i\left(f_{i+1}a^3 - f_i b^3\right) - h_i^2 a^2 b^2 \geq 0\end{array}\right\} \tag{19}$$

where,

$$a = P_i'(0), b = P_i'(1).$$

The constraints on the free parameters can be derived either from Eq. (13) or (14). But Eq. (14) involves a lot of computation, thus we will develop the data dependent constraints for positivity preserving by using Eq. (13) because it is more economy and less intricate as compared with condition (14) and (19).
The sufficient condition in (11) can be rewritten as:

$$\gamma_i = \lambda_i + \text{Max}\left\{0, -\alpha_i\left[\frac{h_i d_i + (2\beta_i + 1) f_i}{f_i}\right], \beta_i\left[\frac{h_i d_{i+1} - (2\alpha_i + 1) f_{i+1}}{f_{i+1}}\right]\right\}, \lambda_i > 0. \tag{20}$$

Remark 1: The positivity-preserving by using the rational spline with $\gamma_i = 0$ have been discussed in details by Hussain and Ali (2006). One of the main differences between the proposed scheme with the work of Hussain and Ali (2006) is that, our rational cubic spline interpolant have two free parameters $\alpha_i > 0$, $\beta_i > 0$ that can be used to refine the final shape of the curves meanwhile there is no free parameters in the work of Hussain and Ali (2006). It is important that, the method should have free parameters in order to modify and altering the final shape of the interpolating curves. Theorem 2 (Hussain and Hussain, 2006) below gives the sufficient condition for positivity preserving by using rational cubic spline of Tian *et al*. (2005). It is the same as our proposed rational cubic with $\gamma_i = 0$.

Theorem 2: (Hussain and Ali, 2006).
The rational cubic spline in (1) with $\gamma_i = 0$ preserves positivity if and only if the following sufficient conditions are satisfied:

$$\alpha_i > \text{Max}\left\{0, -\frac{h_i d_i}{2f_i} + 1\right\}, \beta_i > \text{Max}\left\{0, \frac{h_i d_{i+1}}{2f_i} - 1\right\}. \quad (21)$$

An algorithm to generate C^1 positivity-preserving curves using the results in Theorem 1 is given below.

Algorithm 1:
1. Input the number of data points, n and data points $\{x_i, f_i\}_{i=0}^{n}$.
2. For $i = 0, 1,\ldots, n$ estimate d_i using Arithmetic Mean Method (AMM).
3. For $i = 0, 1,\ldots, n$-1:

- Calculate h_i and Δ_i
- Choose any suitable values of $\alpha_i>0$, $\beta_i>0$
- Calculate the shape parameter γ_i using (20) with suitable choices of $\lambda_i>0$
- Calculate the inner control ordinates A_1 and A_2

4. For $i = 0, 1,\ldots, n$-1

Construct the piecewise positive interpolating curves using (1).
Repeat Step 1 until Step 4 for each tested data sets.

CONSTRAINED DATA MODELING

For data modeling constrained by using the propose rational cubic spline, the result from positivity preserving in Section 3 will be generalized in order to constrained the data that lies above arbitrary straight line y = mx + c. In other word, the sufficient condition for the rational interpolant to be above straight line will be derived. In general, the standard cubic spline interpolation was unable to produce the interpolating curves that will lies above the given straight line. Figure 3 and 4 show these examples. Clearly there are some parts of the curves that lie below the given straight line. This unwanted flaw must be recovered nicely and should be visual pleasing enough for computer graphics displays. The following theorem gives the main results for data constrained modeling by using the proposed rational cubic spline.

Theorem 3: The piecewise rational cubic spline interpolant $s(x)$ preserves the shape of the data that lies above the given straight line $y = mx + c$, if in subinterval $[x_i, x_{i+1}]$, $i = 0, 1,\ldots, n$-1, the free parameter γ_i, satisfy the following sufficient condition:

$$\gamma_i > \text{Max}\left\{0, \frac{\alpha_i\left(-f_i - h_i d_i + b_i\right)}{\left(f_i - a_i\right)}, \frac{\beta_i\left(-f_{i+1} + h_i d_{i+1} + a_i\right)}{f_{i+1} - b_i}\right\}, \quad i = 0,1,\ldots,n-1. \quad (22)$$

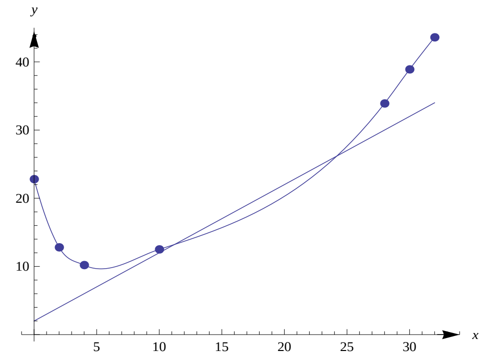

Fig. 3: Default cubic spline polynomial ($\alpha_i = \beta_i = 1$, $\gamma_i = 0$) for data in Table 1

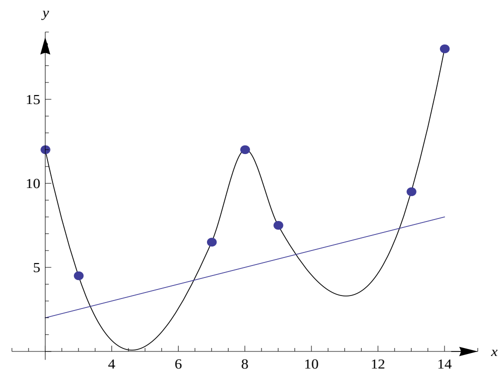

Fig. 4: Default cubic spline polynomial ($\alpha_i = \beta_i = 1$, $\gamma_i = 0$) for data in Table 2

Proof: Assuming that, we are given the set of data (x_i, f_i), $i = 0, 1,\ldots, n$ lying above the straight line y = mx + c, such that:

$$f_i > mx_i + c, i = 0,1,\ldots,n. \quad (23)$$

Here we follow the same idea for data constraint modeling as discussed in details by Shaikh *et al*. (2011) and Sarfraz *et al*. (2013). The curve will lie above the straight y = mx + c. If the proposed rational cubic spline $s(x)$ in (1) satisfies the following condition:

$$s(x) > mx + c, \forall x \in [x_0, x_n]. \quad (24)$$

Now, for each subinterval $[x_i, x_{i+1}]$, $i = 0, 1,\ldots, n$-1, the relation in Eq. (24) can be expressed as follows:

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} > mx_i + c, i = 0,1,\ldots,n \quad (25)$$

By transforming the straight line equation into parametric form, (25) can be rewritten as follows:

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} > (1-\theta)a_i + \theta b_i, i = 0,1,\ldots,n-1 \quad (26)$$

where, $a_i = mx_i + c$, $b_i = mx_{i+1} + c$. Since for all α_i, β_i, γ_i >0, $Q_i(\theta)$ >0, we can multiply the right hand side by denominator in (26). By rearrange (26), the following equation is obtained:

$$S_i(x) = \frac{P_i(\theta) - \{(1-\theta)a_i + \theta b_i\} \times Q_i(\theta)}{Q_i(\theta)} > 0, i = 0,1,\ldots,n-1 \quad (27)$$

Now, we only consider the numerator in (27).
Let:

$$U_i(\theta) = P_i(\theta) - \{(1-\theta)a_i + \theta b_i\} \times Q_i(\theta) = \sum_{j=0}^{3} c_{ij}(1-\theta)^{3-j}\theta^j \quad (28)$$

where,

$$c_{i0} = \alpha_i(f_i - a_i), c_{i1} = (2\alpha_i\beta_i + \alpha_i + \gamma_i)f_i + \alpha_i h_i d_i -$$
$$(2\alpha_i\beta_i + \gamma_i)a_i - \alpha_i b_i, c_{i3} = \beta_i(f_{i+1} - b_i),$$
$$c_{i2} = (2\alpha_i\beta_i + \alpha_i + \gamma_i)f_{i+1} - \beta_i h_i d_{i+1} - (2\alpha_i\beta_i + \gamma_i)b_i - a_i\beta_i.$$

Now, $U_i(\theta)$ >0 if c_{ij}>0. Clearly c_{i0}>0, c_{i3}>0, due to the fact that f_i - a_i>0, f_{i+1} - b_i>0, thus the sufficient condition will be derived from the following conditions:

$$c_{i1} > 0,$$
$$(2\alpha_i\beta_i + \alpha_i + \gamma_i)f_i + \alpha_i h_i d_i - (2\alpha_i\beta_i + \gamma_i)a_i - \alpha_i b_i > 0$$
(29)

and,

$$c_{i2} > 0, (2\alpha_i\beta_i + \beta_i + \gamma_i)f_{i+1} - \beta_i h_i d_{i+1} - (2\alpha_i\beta_i + \gamma_i)b_i - a_i\beta_i > 0 \quad (30)$$

Equation (29) and (30) provides the following relations:

$$\gamma_i > \frac{\alpha_i(-f_i - h_i d_i + b_i)}{f_i - a_i} \quad (31)$$

and,

$$\gamma_i > \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i} \quad (32)$$

The sufficient conditions in (31) and (32) can be rewrite as follows:

$$\gamma_i > \text{Max}\left\{0, \frac{\alpha_i(-f_i - h_i d_i + b_i)}{(f_i - a_i)}, \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i}\right\}, \ i = 0,1,\ldots,n-1. \quad (33)$$

The result is obtained as require.

The sufficient conditions in Eq. (22) can be rewritten as follows:

$$\gamma_i = \upsilon_i + \text{Max}\left\{0, \frac{\alpha_i(-f_i - h_i d_i + b_i)}{(f_i - a_i)}, \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i}\right\}, \ i = 0,1,\ldots,n-1. \quad (34)$$

where υ_i>0.

The sufficient condition in (34) will guarantee the existence of rational cubic interpolant that lies above the straight line. For the purpose of numerical comparison later, Theorem 4 (Hussain and Hussain, 2006) below gives the sufficient condition for data constrained modeling by using Hussain and Hussain (2006) method. It is equivalent with our proposed rational cubic with $\gamma_i = 0$.

Theorem 4: (Hussain and Hussain, 2006).

The rational cubic spline in (1) with $\gamma_i = 0$ preserves the shape of the data lies above the straight line, if in subinterval $[x_i, x_{i+1}]$, free parameters α_i and β_i satisfy the following sufficient conditions:

$$\alpha_i > \text{Max}\left\{0, \frac{-f_{i+1} + h_i d_{i+1} + a_i}{2(f_{i+1} - b_i)}\right\}, \beta_i > \text{Max}\left\{0, \frac{-f_i - h_i d_i + b_i}{2(f_i - a_i)}\right\}. \quad (35)$$

Algorithm 2 can be used in order to generate the interpolating curves that lies above any straight line provided that the free parameter γ_i must be chosen from (34) with some suitable values of the other there parameters α_i, β_i and υ_i.

Algorithm 2:

1. Input the number of data points, n, data points $\{x_i, f_i\}_{i=0}^{n}$, and straight line equation, $y = mx + c$.
2. For i = 0, 1,…, n, estimate d_i using Arithmetic Mean Method (AMM):
3. For i = 0, 1,…, n-1:

- Calculate h_i and Δ_i
- Transform the straight line equation into parametric form and calculate a_i and b_i
- Choose any suitable values of α_i>0, β_i>0
- Calculate the shape parameter γ_i using (34) with suitable choices of υ_i>0
- Calculate the inner control ordinates A_1 and A_2 defined in (1)

4. For i = 0, 1,…, n-1

Construct the piecewise data constrained modeling interpolating curves using (1).

Table 1: A data from Hussain and Hussain (2006) above y = x + 2

x_i	0	2	4	10	28	30	32
f_i	22.80	12.80	10.2000	12.5000	33.900	38.900	43.600
d_i	-6.85	-3.15	-0.8792	0.5847	2.369	2.425	2.275

Table 2: A data from Hussain and Hussain (2006) above y = x/2 + 1

x_i	2	3	7	8	9	13	14
f_i	12	4.5	6.5	12	7.5	9.5	18
d_i	-9.1	-5.9	4.5	0.5	-3.5	6.9	10.1

Table 3: A positive data from Brodlie and Butt (1991)

x_i	0	2	4	10	28	30	32
f_i	20.80	8.80	4.2000	0.5000	3.9000	6.200	9.600
d_i	-7.85	-4.15	-1.8792	-0.4153	1.0539	1.425	1.975

Table 4: A positive data from Sarfraz *et al*. (2005)

x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	3	10
d_i	-9.65	-6.35	3.25	0	-3.95	5.65	8.35

Remark 2: The case for constrained data modeling that lies below arbitrary straight line can be treated in the same manner.

RESULTS AND DISCUSSION

In this subsection the positivity preserving and data constrained modeling by using the results obtained in Theorem 1 and Theorem 3 are explored. Two sets of positive data taken from Brodlie and Butt (1991) and Sarfraz *et al*. (2005) were used. Meanwhile Table 1 and 2 show the data that lies above the straight line taken from Hussain and Hussain (2006).

Figure 1 and 2 shows the default cubic spline interpolation for data in Table 3 and 4 respectively. Figure 5 and 6 show the examples of positivity preserving by using Hussain and Ali (2006). Figure 7 to 9, show the shape preserving by using our propose rational cubic spline scehemes with various choices of free parameters α_i, β_i with λ_i = 0.5. Meanwhile Figure 10 shows the positivity preserving by using our propose scehemes (blue) and Hussain and Ali (2006) (red) for data in Table 3. Figure 11 to 13, show the positivity preserving by using our propose rational cubic spline with various choices of shape parameters α_i, β_i with λ_i = 0.5 Figure 14 shows visual pleasing positive interpolating curves with α_i = 0.5, 2, 0.5, 0.5, 3.5, 0.5, β_i = 2, 8, 1, 1.5, 5.45, 0.8 for in Table 4. Figure 15 shows shape preserving by using our schemes (red) with $\alpha_i = \beta_i = 0.5$, i = 1, 2,…, 5 and $\alpha_0 = \beta_0 = 2.5$ and Hussain and Ali (2006) (blue).

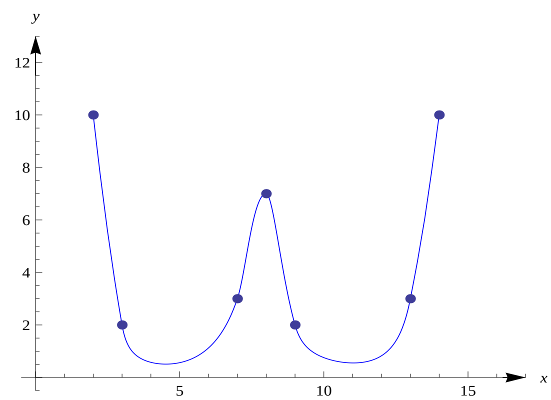

Fig. 6: Shape preserving using Hussain and Ali (2006) for data in Table 4

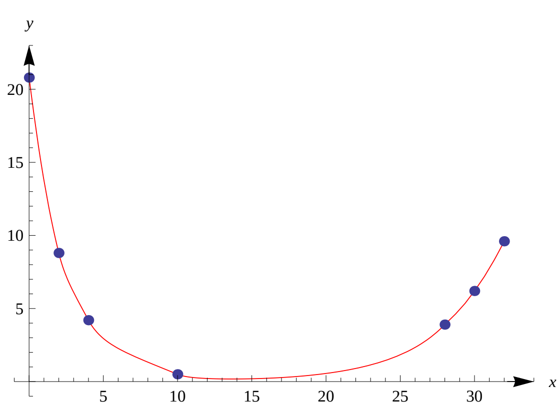

Fig. 5: Shape preserving using Hussain and Ali (2006) for data in Table 3

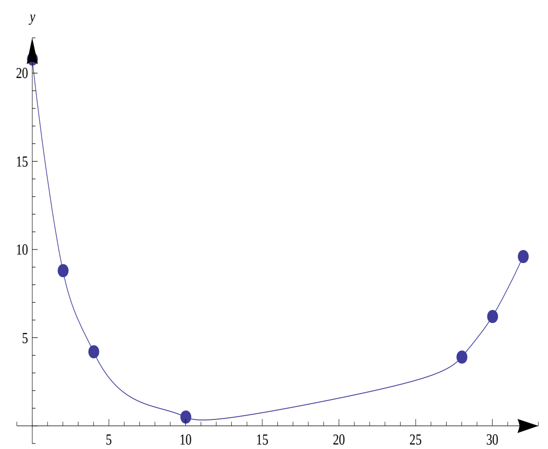

Fig. 7: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 1$) for data in Table 3

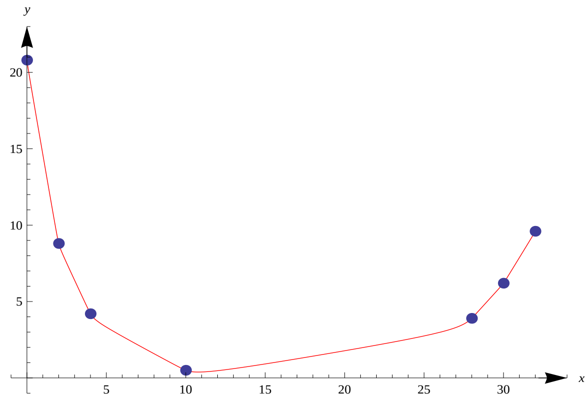

Fig. 8: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 0.01$) for data in Table 3 (the rational spline approach to straight line)

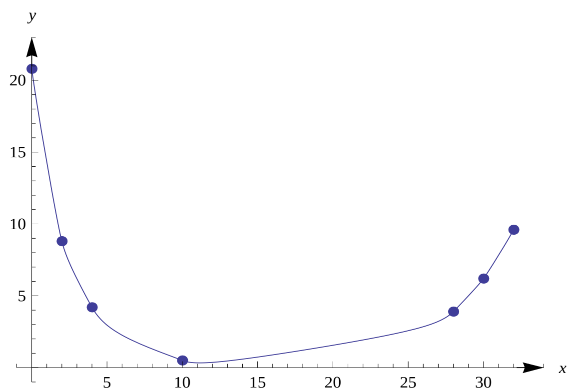

Fig. 9: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 0.5$) for data in Table 3

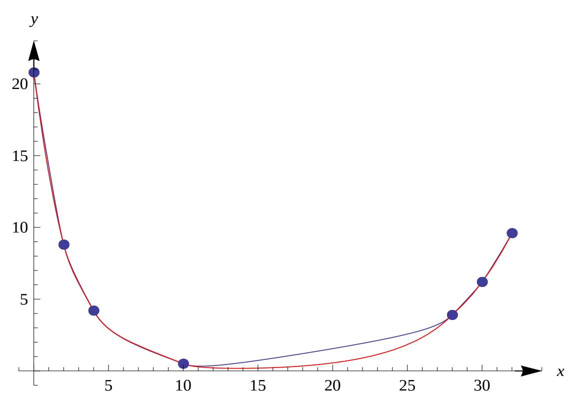

Fig. 10: Shape preserving with $\alpha_i = \beta_i = 2.5$ (blue) and Hussain and Ali (2006) (red) for data in Table 3

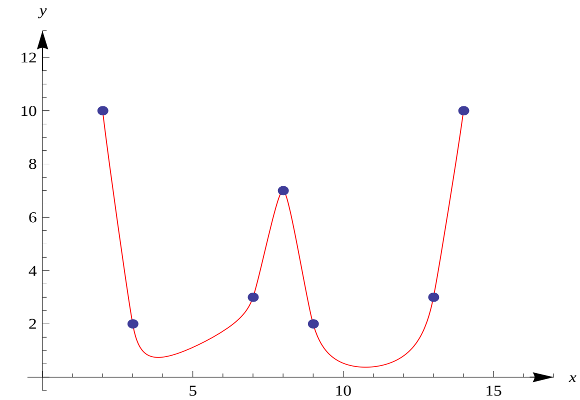

Fig. 11: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 1$) for data in Table 4

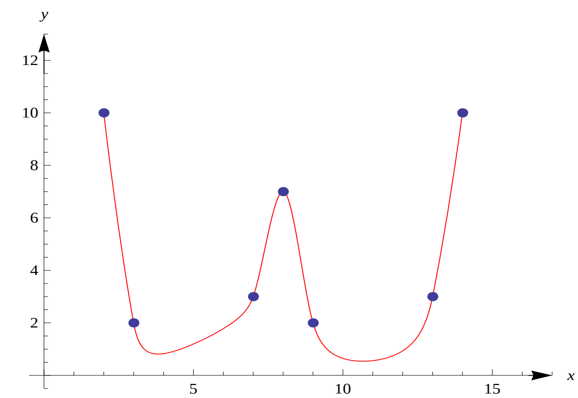

Fig. 12: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 0.5$) for data in Table 4

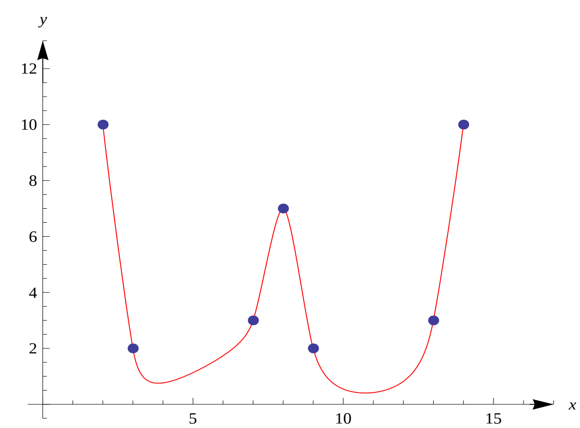

Fig. 13: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 1.5$) for data in Table 4

Figure 3 and 4, show the default cubic spline constrained interpolation for data in Table 1 and 2 respectively. For data constrained modeling, Figure 16 and 17 show the shape preserving by using Hussain and Hussain (2006) method. Figure 18 to 20 show the examples of shape preserving by using our propose rational cubic with various choices of shape parameters as indicated in the respective figures. Figure 21 shows the examples on shape control analysis for data in Table 2. It can be seen clearly that, for the fixed values of α_i, β_i and keep changing the γ_i values, the rational cubic spline interpolating curves will lies above the given straight line. From Fig. 21, when $\alpha_i = \beta_i = 1$ and $\gamma_i = 8$ (shown in blue color) the curves lies above the

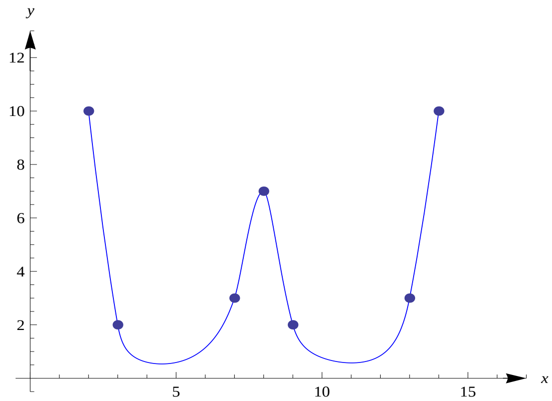

Fig. 14: Shape preserving using our proposed rational spline for data in Table 4 (visual pleasing)

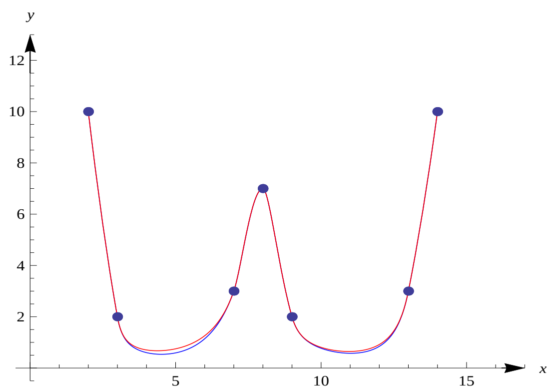

Fig. 15: Shape preserving interpolation with our proposed spline (red) and Hussain and Ali (2006) (blue) for data in Table 4

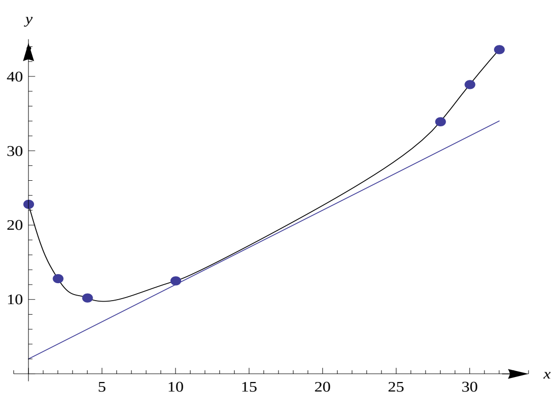

Fig. 16: Shape preserving using Hussain and Hussain (2006) for data in Table 1

straight line. Figure 22 and 23 show the shape preserving by using rational cubic spline with $\upsilon_i = 0.25$. Figure 24 shows two curves both lies above straight line with $\alpha_i = \beta_i = 0.5$, $\gamma_i = 8$ (black) and $\alpha_i = \beta_i = 0.5$, $\gamma_i = 5$ (blue). Finally Fig. 25 shows interpolating curves for Hussain and Hussain (2006) (black) and our rational scheme (red).

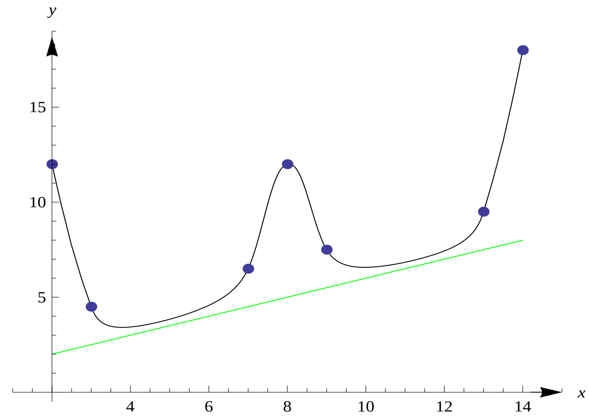

Fig. 17: Shape preserving using Hussain and Hussain (2006) for data in Table 2

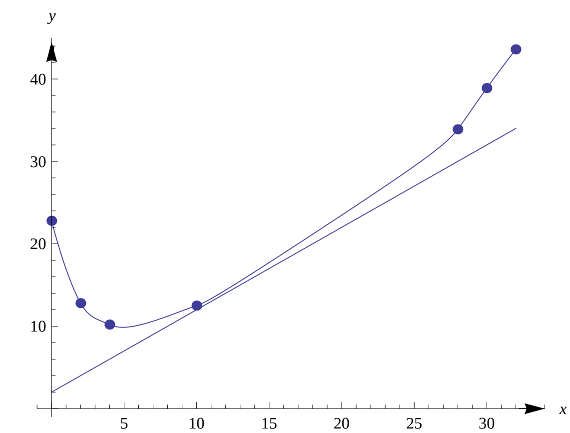

Fig. 18: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 1$, $\upsilon_i = 0.25$) for data in Table 1

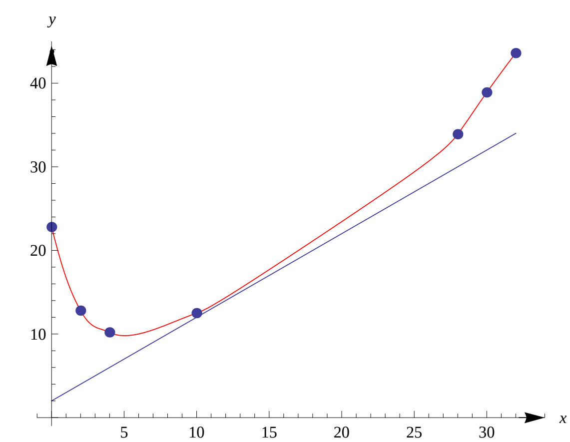

Fig. 19: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 0.5$, $\upsilon_i = 0.25$) for data in Table 1

From all 25 figures that shown in this section, it is clear that, the proposed rational cubic spline with three parameters provides greater flexibility in controlling the final shape of positive and constraint interpolating curves. The positive and constraint interpolating curves

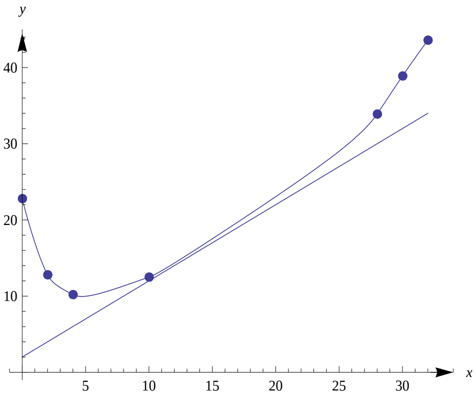

Fig. 20: Shape preserving using our proposed rational spline with ($\alpha_i = 1$, $\beta_i = 2$, $\upsilon_i = 0.1$) for data in Table 1

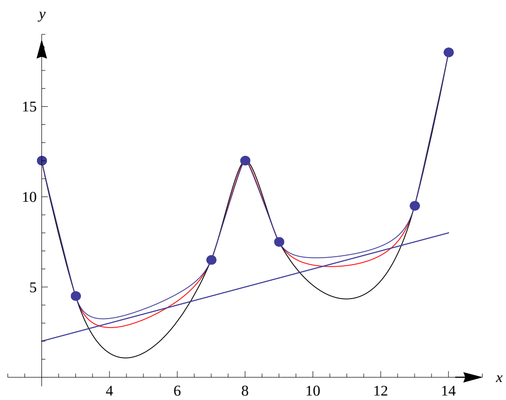

Fig. 21: Rational spline with $\alpha_i = \beta_i = \gamma_i = 1$ (black), $\alpha_i = \beta_i = 1$, $\gamma_i = 5$ (red), $\alpha_i = \beta_i = 1$, $\gamma_i = 8$ (blue) for data in Table 2

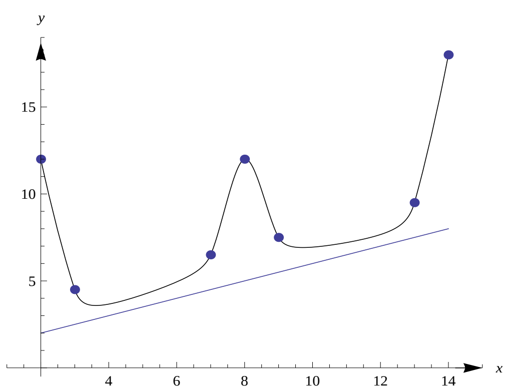

Fig. 22: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 1$, $\upsilon_i = 0.25$) for data in Table 2

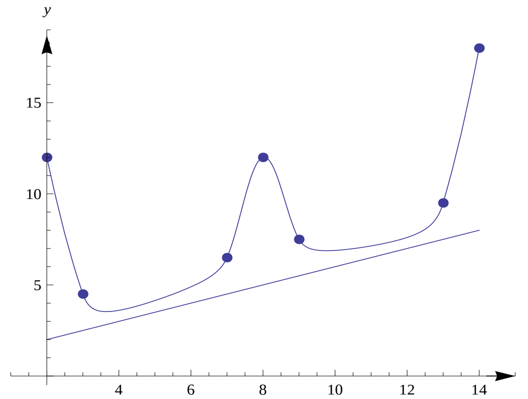

Fig. 23: Shape preserving using our proposed rational spline with ($\alpha_i = \beta_i = 0.5$, $\upsilon_i = 0.25$) for data in Table 2

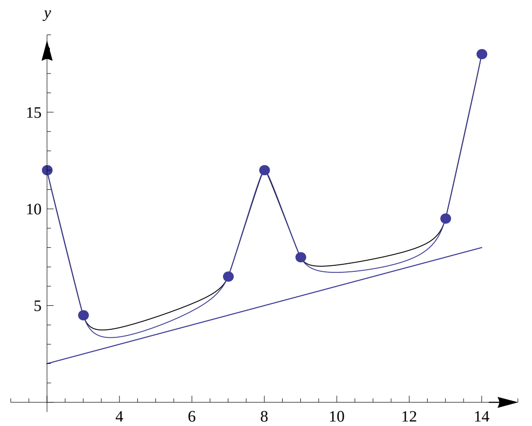

Fig. 24: Shape preserving using our proposed rational spline with $\alpha_i = \beta_i = 0.5$, $\gamma_i = 8$ (black) and $\alpha_i = \beta_i = 0.5$, $\gamma_i = 5$ (blue) for data in Table 2

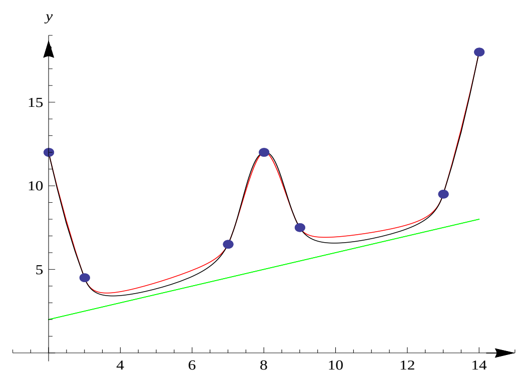

Fig. 25: Shape preserving spline with $\alpha_i = \beta_i = 1$, $\upsilon_i = 0.25$ (red) and Hussain and Hussain (2006) (black) for data in Table 2

can be change by choosing different values of two free parameters α_i, β_i and the sufficient conditions is derived on the other parameter γ_i. Theorem 1 and Theorem 3 will guarantee the existence of positive and constrained data interpolation curves respectively for $\alpha_i, \beta_i>0$, $\gamma_i \geq 0$. Once we assigned the values of α_i, β_i, then by choosing some suitable values of $\lambda_i>0$ (for positivity) and $\upsilon_i>0$ (for data constrained), the value of γ_i can be calculated from Eq. (20) and (34), respectively. This process can be repeated to obtain the desire interpolating curves as user wish.

From numerical comparison between our proposed rational cubic spline with the works of (1) Hussain and

Ali (2006) for positivity preserving and (2) Hussain and Hussain (2006) for data constrained modeling (lies above arbitrary straight line), it is clear that our proposed rational cubic spline gives a good results and it is comparable with both methods. It also provides good alternative to the existing rational cubic spline for positivity and constrained data modeling. Free parameters α_i and β_i can be utilized in order to change the final shape of the curves. This will gives us an added value to the proposed rational cubic spline scheme. Furthermore, the proposed rational cubic spline (cubic/quadratic) with three parameters can be extended to preserves the positive data with C^2 continuity. The authors in the final process to complete the study.

Final remark: Even though the proposed rational cubic spline with three parameters is a minor variant of the work of Tian *et al*. (2005), Hussain and Ali (2006) and Hussain and Hussain (2006), the free parameters α_i and β_i provide the user in controlling the final shape of positive curves and data constrained curves. In the original works of Hussain and Ali (2006) and Hussain and Hussain (2006), there is no free parameter.

CONCLUSION

New rational cubic spline with three parameters has been discussed in details in this work. Firstly, the proposed rational interpolant has been use for positivity preserving. While the sufficient condition for positivity has been derived on one of the parameter γ_i and the other two parameters α_i and β_i will be acting as free parameter. The free parameter can be used to refine and modify the final shape of the interpolating positive curves. Secondly, the results for positivity preserving has been extended to study the data constrained modeling by using the proposed rational cubic spline. Theorem 1 and 3 will guarantee the existence of positive and constrained data interpolation curves respectively. Furthermore when α_i, β_i>0 and γ_i = 0 the proposed scheme reduces to the work of Hussain and Ali (2006) and Hussain and Hussain (2006) for positivity preserving and constrained data interpolation respectively. One of the main advantages of the proposed scheme is that there exists two free parameters α_i and β_i compare to the work by Hussain and Ali (2006) and Hussain and Hussain (2006) have no free parameter (s). Works on preserving the monotone data as well as the convex data are underway and it is interesting to use this method for medical image processing. Another potential research study is to use the rational cubic spline for surface interpolating for positive, monotonicity and convexity data. This will be our main target for future research.

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