Research Article
Bayes Estimation for Inverse Rayleigh Model under Different Loss Functions

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Abstract: The inverse Rayleigh distribution plays an important role in life test and reliability domain. The aim of this article is study the Bayes estimation of parameter of inverse Rayleigh distribution. Bayes estimators are obtained under squared error loss, LINEX loss and entropy loss functions on the basis of quasi-prior distribution. Comparisons in terms of risks with the estimators of parameter under three loss functions are also studied. Finally, a numerical example is used to illustrate the results.

Keywords: Bayes estimator, entropy loss, LINEX loss, risk function, squared error loss

INTRODUCTION

The inverse Rayleigh distribution is one of the most important lifetime distributions. It has many applications in the area of reliability and life testing study. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution. The statistical inference for the inverse Rayleigh distribution has drawn great attention by many authors. For example, Voda (1972) presented some properties of the maximum likelihood estimator, for inverse Rayleigh distribution, furthermore confidence intervals and tests of hypotheses are developed. Gharraph (1993) derived five measures of location for the inverse Rayleigh distribution, These measures are the mean, harmonic mean, geometric mean, mode and the median. Abdel-Monem (2003) developed some estimation and prediction results for the inverse Rayleigh distribution. Al-Hussaini and Ahmed (2003) studied the Bayesian prediction bounds for the sth future record value. El-Helbawy and Abd-El-Monem (2005) obtained Bayesian estimators and one and two sample predictions of the parameter of the inverse Rayleigh distribution under for different loss functions. Soliman *et al.* (2010) discussed the Bayes estimation and prediction for inverse Rayleigh distribution based on lower record values. Dey (2012) studied the Bayes estimation of the parameter of inverse Rayleigh distribution under squared error loss and LINEX loss functions. These articles have done good work on Rayleigh distribution, but how to choose the best Bayesian estimators is still not given. Thus, this article will discuss the Bayes estimation of the parameter of inverse Rayleigh distribution under three loss functions and will give a rule for the selection of best Bayes estimators.

Suppose X is a random variable from inverse Rayleigh distribution if the probability density function is given by:

$$f(x;\theta) = \frac{2}{\theta x^3}\exp(-\frac{1}{\theta x^2}), \quad x > 0 \tag{1}$$

where, $\theta > 0$ is unknown parameter.

METHODOLOGY

Maximum likelihood estimation: Suppose $X_1, X_2,\ldots, X_n$ are n samples from the inverse Rayleigh distribution (1) and $x = (x_1, x_2,\ldots, x_n)$ is the observation of $X = (X_1, X_2,\ldots, X_n)$. The Likelihood Function (LF) is given by:

$$l(\theta;x) = \frac{1}{\theta^n}\prod_{i=1}^{n}\frac{2}{x_i^3}e^{-\frac{s}{\theta}} \tag{2}$$

where, $s = \sum_{i=1}^{n}\frac{1}{x_i^2}$ is the observation of $S = \sum_{i=1}^{n}\frac{1}{x_i^2}$.

The maximum likelihood estimator of θ is easily derived as:

$$\hat{\theta}_{ML} = \frac{S}{n} \tag{3}$$

And it can be easily shown that $S = \sum_{i=1}^{n}\frac{1}{x_i^2}$ is distributed the Gamma distribution $\Gamma(n,\theta^{-1})$.

Bayes estimation: Suppose that some prior knowledge about the parameter θ is available to the investigation from past experience with the inverse Rayleigh model. The prior knowledge can often be summarized in terms of the so-called prior densities on parameter space of θ. For the situation where the experimenter has no prior

information about the parameter θ, one may use the quasi density as given by:

$$\pi(\theta;d) \propto \frac{1}{\theta^d}, \quad \theta > 0, d > 0 \tag{4}$$

Hence, $d = 0$ leads to a diffuse prior and $d = 1$ to a non-informative prior.

In Bayesian estimation, an important element is the selection of a loss function $L(\hat{\theta}, \theta)$, where $\hat{\theta}$ is a decision rule based on the data. The squared error loss as the most common symmetric loss function is widely used due to its great analysis properties. And the Squared Error Loss Function (SELF) is given as:

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \tag{5}$$

It is a symmetrical loss function that assigns equal losses to overestimation and underestimation. However, in many practical problems, Basu and Ebrahimi (1992) pointed that overestimation and underestimation will make different consequents. Thus using of the symmetric loss function may be inappropriate and to overcome this difficulty, many asymmetric loss functions are put forward. The LINEX loss is one of the most used.

The LINEX loss function: Varian (1975) proposed an asymmetric loss function known as the LINEX loss function and Zellner (1986) applied it to Bayes estimation and prediction problems. The LINEX loss function is suitable for situations where overestimation is more costly than underestimation. When estimating a parameter θ by $\hat{\theta}$, the LINEX is given by Basu and Ebrahimi (1992).

$$L(\Delta) = e^{a\Delta} - a\Delta - 1, \quad a \neq 0 \tag{6}$$

where $\Delta = \frac{\hat{\theta}}{\theta} - 1$. The Bayes estimator of θ, denoted by $\hat{\theta}_{BL}$ under the LINEX loss function is given by the equation:

$$E_\pi[\frac{1}{\theta}\exp(\frac{a\hat{\theta}_{BL}}{\theta})] = e^a E_\pi[\frac{1}{\theta}] \tag{7}$$

Provide that the posterior expectation $E_\pi(\cdot)$ in Eq. (7) exists and is finite.

The entropy loss function: In many practical situations, it appears to be more suitable to express the loss in terms of the ratio $\hat{\theta}/\theta$. In this case, Dey *et al.* (1987) pointed out that a useful asymmetric loss function is entropy loss function:

$$L(\hat{\theta}, \theta) = \frac{\hat{\theta}}{\theta} - \ln\frac{\hat{\theta}}{\theta} - 1 \tag{8}$$

Whose minimum occurs at $\hat{\theta} = \theta$. Also, this loss function has been used in Dey and Liu (1992) and Calabria and Pulcini (1994). The Bayes estimator under the entropy loss is denoted by $\hat{\theta}_{BE}$, given by:

$$\hat{\theta}_{BE} = [E_\pi(\theta^{-1})]^{-1} \tag{9}$$

Combining (2) with quasi-prior (4) using Bayes theorem, the posterior pdf of θ is given by:

$$h(\theta \mid x) = \frac{s^{n+d-1}}{\Gamma(n+d-1)}\theta^{-(n+d)}e^{-\frac{s}{\theta}} \tag{10}$$

Then:

- The Bayes estimator under the squared error loss function is given by:

$$\hat{\theta}_{BS} = \frac{S}{n+d-2}, \quad n+d > 2 \tag{11}$$

- Using (10), the Bayes estimator under the LINEX loss function is come out to be:

$$\hat{\theta}_{BL} = \frac{1 - \exp(-\frac{a}{n+d})}{a}S \tag{12}$$

- Using (12), the Bayes estimator under the entropy loss function is obtained as:

$$\hat{\theta}_{BE} = \frac{S}{n+d-1} \tag{13}$$

Risk function: The risk functions of the estimators $\hat{\theta}_{BS}, \hat{\theta}_{BL}$ and $\hat{\theta}_{BE}$ relative to Squared error loss (5) are denoted by $R(\hat{\theta}_{BS})$, $R(\hat{\theta}_{BL})$ and $R(\hat{\theta}_{BE})$, respectively, are can be easily shown as:

$$R(\hat{\theta}_{BS}) = \theta^2\left[\frac{n(n+1)}{(n+d-2)^2} - \frac{2n}{n+d-2} + 1\right]$$

$$R(\hat{\theta}_{BL}) = \theta^2\left[\frac{n(n+1)}{a^2}(1 - e^{-a/(n+d)})^2 - \frac{2n}{a}(1 - e^{-a/(n+d)}) + 1\right]$$

$$R(\hat{\theta}_{BE}) = \theta^2\left[\frac{n(n+1)}{(n+d-1)^2} - \frac{2n}{n+d-1} + 1\right]$$

Figure 1 to 4 have plotted the ratio of the risk functions to θ^2, i.e.:

$$\frac{R(\hat{\theta}_{BS})}{\theta^2} = B_1, \quad \frac{R(\hat{\theta}_{BL})}{\theta^2} = B_2, \text{ and } \frac{R(\hat{\theta}_{BE})}{\theta^2} = B_3$$

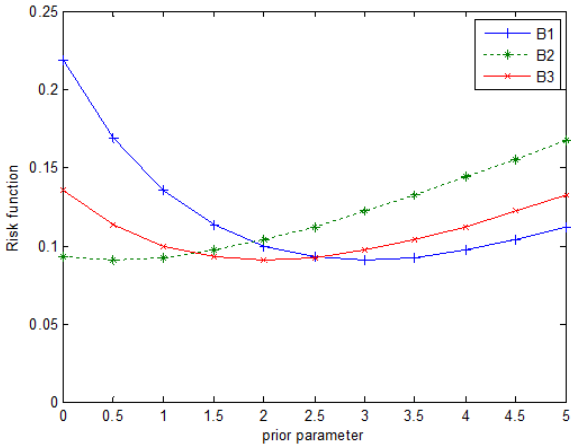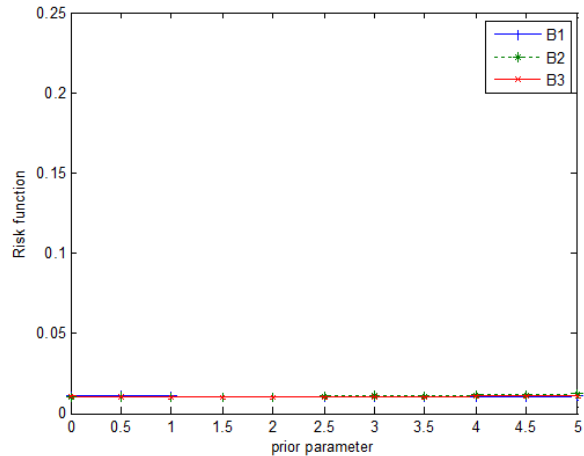Fig. 1: Ratio of the risk functions with n = 10
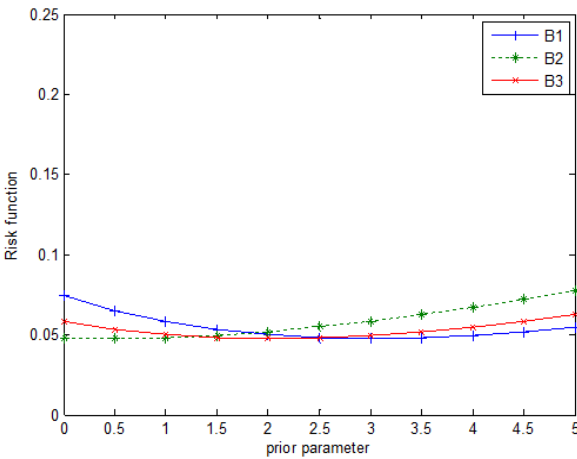

Fig. 2: Ratio of the risk functions with n = 20

Fig. 3: Ratio of the risk functions with n = 50
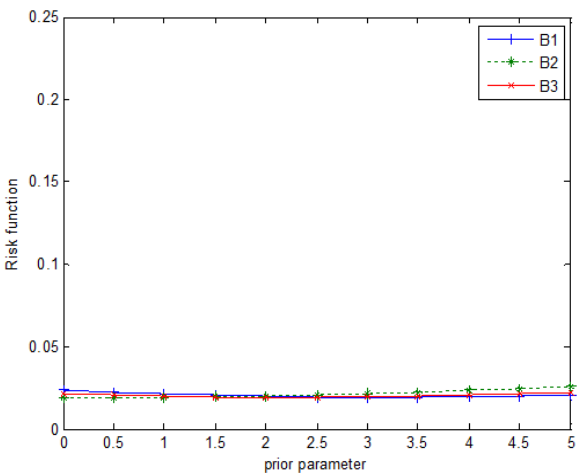

Fig. 4: Ratio of the risk functions with n = 100

distribution and this choice in return depends on the situation at hand. But when n is large (n>50), the three ratios of the risk functions are almost equal, so when the sample size is large, any estimator can be chosen for practical use.

EMPIRICAL STUDY

In this section, a Monte Carlo simulation is used to compare the estimators obtained in this study. The MLE, Bayes estimators of the parameter θ are computed according to the following steps:

Step 1: For given values of θ, a sample of size n is then generated from the density of the inverse Rayleigh distribution (1), which is considered to be the informative sample.
Step 2: The MLE and Bayes estimators is calculated based on above section.
Step 3: Steps1-2 are repeated 1000 times and the risks under squared-error loss of the estimates, noted by ER, are computed by using:

$$ER(\hat{\theta}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2$$

where, $\hat{\theta}_i$ is the estimate at the ith run.

Numerical simulation expressed in Table 1 shows that though under small sample sizes, the risks under squared-error loss of the estimates have big differences, but when the sample sizes are large (n>50), they are almost equal, thus any of the three Bayes estimators be chosen in practical application.

CONCLUSION

The inverse Rayleigh distribution plays an important role in life test and reliability domain. Thus

From Fig. 1 to 4, it is clear that no of the estimators uniformly dominates any other. Therefore these estimators can be chosen according to the value of d when the quasi-prior density is used as the prior

Table 1: Estimated value (Estimated Risk (ER)) for different sample sizes

n	$\hat{\theta}_{ML}$	d = 1.0, a = 1.0			d = 1.5, a = 1.0		
		$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$\hat{\theta}_{BE}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$\hat{\theta}_{BE}$
20	0.9942	1.0465	0.9246	0.9942	1.0197	0.9036	0.9699
	(0.0500)	(0.0575)	(0.0489)	(0.0500)	(0.0530)	(0.0506)	(0.0485)
30	1.0076	1.0424	0.9596	1.0076	1.0247	0.9446	0.9911
	(0.0326)	(0.0366)	(0.0311)	(0.0326)	(0.0342)	(0.0316)	(0.0315)
40	1.0037	1.0215	0.9674	1.0037	1.0164	0.9559	0.9913
	(0.0252)	(0.0273)	(0.0244)	(0.0252)	(0.0261)	(0.0248)	(0.0246)
50	1.0000	1.0204	0.9709	1.0000	1.0101	0.9605	0.9901
	(0.0191)	(0.0203)	(0.0189)	(0.0191)	(0.0196)	(0.0192)	(0.0183)
75	0.9999	1.0134	0.9803	0.9999	1.0066	0.9739	0.9933
	(0.0129)	(0.0134)	(0.0128)	(0.0129)	(0.0131)	(0.0129)	(0.0128)
100	0.9939	1.0040	0.9792	0.9939	0.9989	0.9744	0.9890
	(0.0096)	(0.0100)	(0.0099)	(0.0098)	(0.0098)	(0.0100)	(0.0098)
150	0.9945	1.0012	0.9846	0.9945	0.9978	0.9814	0.9912
	(0.0066)	(0.0067)	(0.0067)	(0.0066)	(0.0066)	(0.0067)	(0.0066)
200	0.9984	1.0034	0.9909	0.9984	1.0009	0.9885	0.9959
	(0.0051)	(0.0051)	(0.0051)	(0.0051)	(0.0051)	(0.0051)	(0.0051)

this article studied the Bayes estimation of parameter of inverse Rayleigh distribution. Three Bayes estimators are obtained under squared error loss, LINEX loss and entropy loss functions. The best Bayes estimator are chosen by comparisons in terms of risks with the estimators and numerical simulation are also shown that when the sample sizes are large (n>50), these estimators are almost equal, thus any of the three Bayes estimators be chosen in practical application. The result can also used other distributions, such as exponential, Weibull, Rayleigh distributions.

REFERENCES

Abdel-Monem, A.A., 2003. Estimation and prediction for the inverse Rayliegh life distribution. M.Sc. Thesis, Faculty of Education, Ain Shames University.

Al-Hussaini, E.K. and A.B.A. Ahmed, 2003. On Bayesian interval prediction of future records. Test, 12: 79-99.

Basu, A.P. and N. Ebrahimi, 1992. Bayesian approach to life testing and reliability estimation using asymmetric loss function. J. Stat. Plan. Infer., 29: 21-31.

Calabria, R. and G. Pulcini, 1994. An engineering approach to Bayes estimation for the Weibull distribution. Micro Electron. Reliab., 34: 789-802.

Dey, S., 2012. Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution. Malays. J. Math. Sci., 6: 113-124.

Dey, D.K. and P.S.L. Liu, 1992. On comparison of estimators in a generalized life model. Microelectron. Reliab., 32: 207-221.

Dey, D.K., M. Ghosh and C. Srinivasan, 1987. Simultaneous estimation of parameters under entropy loss. J. Stat. Plan. Infer., 15: 347-363.

El-Helbawy, A.A. and Abd-El-Monem, 2005. Bayesian estimation and prediction for the inverse Rayleigh lifetime distribution. Proceeding of the 40th Annual Conference of Statistics, Computer Sciences and Operation Research, ISSR, Cairo University, Eygpt, pp: 45-59.

Gharraph, M.K., 1993. Comparison of estimators of location measures of an inverse Rayleigh distribution. Egypt. Stat. J., 37: 295-309.

Soliman, A., E.A. Amin and A.A. Abd-ElAziz, 2010. Estimation and prediction from inverse Rayleigh distribution based on lower record values. Appl. Math. Sci., 4: 3057-3066.

Varian, H.R., 1975. A Bayesian Approach to Real Estate Assessment. In: Fienberg, S.E. and A. Zellner (Eds.), Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage. North Holland, Amsterdam, pp: 195-208.

Voda, R.G., 1972. On the inverse Rayleigh variable. Rep. Stat. Apph. Res. Juse, 19: 15-21.

Zellner, A., 1986. Bayesian estimation and prediction using asymmetric loss functions. J. Am. Stat. Assoc., 81: 446-451.