

Research Article

Bayes Estimation of Shape Parameter of Minimax Distribution under Different Loss Functions

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Abstract: The object of this study is to study the Bayes estimation of the unknown shape parameter of Minimax distribution. The prior distribution used here is the non-informative quasi-prior of the parameter. Bayes estimators are derived under squared error loss function and three asymmetric loss functions, which are the LINEX loss, precaution loss and entropy loss functions. Monte Carlo simulations are performed to compare the performances of these Bayes estimates under different situations. Finally, we summarize the result and give the conclusion of this study.

Keywords: Bayes estimator, entropy loss, LINEX loss, minimax distribution, precautionary loss, squared error loss

INTRODUCTION

In lifetime and reliability field, there are many distributions such as exponential, Weibull, Gamma and inverse Gamma distribution used in modeling lifetime model. Despite the many alternatives and generalizations (Kotz and Van Dorp, 2004; Nadarajah and Gupta, 2004), it remains fair to say that the Beta distribution provides the premier family of continuous distributions on bounded support (which is taken to be interval (0, 1)). The beta distribution with two shape parameters a and b is denoted by Beta (a, b) and it has following probability density function (pdf):

$$g(x; a, b) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1$$

Here a and b are positive real numbers and $B(\cdot, \cdot)$ is the Beta function. The shapes of the pdf of Beta (a, b) are unimodal, uniantimodal, increasing, decreasing or constant depending on the values of a and b relative to 1 and have a host of other attractive properties (Johnson *et al.*, 1994).

Minimax distribution is a special case of generalized beta distribution, which firstly proposed by McDonald (1984). Jones (2007) suggested an alternative two-parameter distribution on (0, 1), which has many of the same properties as the Beta distribution but has some advantages in terms of tractability. He called this distribution Minimax distribution and termed it as Minimax (α, θ), where its two shape parameters α and θ are positive. Let X is a random variable distributed with Minimax (α, θ), the corresponding probability density function (pdf) and cumulative distribution function (cdf) of a Minimax distributed are respectively given by:

$$f(x; \theta, \alpha) = \frac{\alpha}{\theta} x^{\alpha-1} (1-x^\alpha)^{\frac{1}{\theta}-1}, \quad x > 0 \tag{1}$$

and,

$$F(x; \theta, \alpha) = 1 - (1-x^\alpha)^{\frac{1}{\theta}}, \quad x > 0 \tag{2}$$

where, $\theta > 0$ and $\alpha > 0$ are shape parameters, respectively. Here we assume that the parameter α is known.

For Minimax distribution, Shadrokh and Pazira (2010) study the minimax estimation on the Minimax distribution; Makhdoom (2011) and Deiri (2011) are respectively obtained the maximum likelihood and moment estimators of the parameters of the Minimax distribution in the presence of one outlier and two outlier.

This paper will study the Bayes estimation of the shape parameter θ of Minimax (α, θ) under various loss functions on the basis of the Gamma prior distribution of shape parameter.

PRELIMINARIES KNOWLEDGE

In Bayesian statistics analysis, loss function and prior distribution are two important elements. In this study, we suppose the prior distribution of the shape parameter θ is non-informative quasi-prior distribution with respect to the following form:

$$\pi(\theta; d) \propto \frac{1}{\theta^d}, \quad \theta > 0, d > 0 \tag{3}$$

Here, $d = 0$ leads to a diffuse prior and $d = 1$ to a non-informative prior.

The other important element of Bayesian analysis is the selection of a loss function, $L(\hat{\theta}, \theta)$, where $\hat{\theta}$ is a decision rule based on the data. One disadvantage of squared error loss is that it assigns equal losses to overestimation and underestimation. Overestimation of a parameter can lead to more severe or less severe consequences than underestimation, or vice versa. Subsequently, an asymmetrical loss function can associate greater importance to overestimation or underestimation and thus it can be as an alternative of symmetric loss function. Several useful asymmetric loss functions are studied and proposed by many authors (Varian, 1975; Zellner, 1986; Basu and Ebrahimi, 1991; Norstorm, 1996; Dey *et al.*, 1987). In this study, the three asymmetric loss functions are used as follows.

LINEX loss function: For the reason of using of the symmetric loss function may be inappropriate in some cases, especially in estimation of reliability, Varian (1975) firstly proposed the LINEX loss function and then it is developed and used in Bayes analysis by Zellner (1986). LINEX loss function has become the mostly important asymmetric function, which is suitable for situations where overestimation is more costly than underestimation. When estimating a parameter θ by $\hat{\theta}$, the LINEX loss function is given by Basu and Ebrahimi (1991):

$$L(\Delta) = e^{c\Delta} - c\Delta - 1, \quad c \neq 0 \tag{4}$$

where, $\Delta = \frac{\hat{\theta}}{\theta} - 1$.

Under the LINEX loss function (4), the Bayes estimator of θ , denoted by $\hat{\theta}_{BL}$ is the solution of the following equation:

$$E\left[\frac{1}{\theta} \exp\left(\frac{c\hat{\theta}_{BL}}{\theta} \mid X\right)\right] = e^c E\left[\frac{1}{\theta} \mid X\right] \tag{5}$$

Provide that the posterior expectation $E(\cdot \mid X)$ in Eq. (5) exists and is finite.

Precautionary loss function: A very useful and simple asymmetric precaution loss function proposed by Norstorm (1996), is given by:

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}} \tag{6}$$

The loss function (6) infinitely near to the origin and thus it can prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimator. It is very useful when underestimation may lead to serious consequences. This loss function was used by several authors

(Yarmohammadi and Pazira, 2010; Pandey and Rao, 2009).

Under the precautionary loss function (6), the Bayes estimator, denoted by $\hat{\theta}_{BP}$ can be given by:

$$\hat{\theta}_{BP} = [E(\theta^2 \mid X)]^{1/2} \tag{7}$$

Entropy loss function: In many practical situations, it appears to be more realistic to express the loss in term of the ratio $\hat{\theta}/\theta$. In this case, another useful asymmetric loss function is the entropy loss function (Dey *et al.*, 1987) with respect to the following form:

$$L(\hat{\theta}, \theta) \propto \frac{\hat{\theta}}{\theta} - \ln \frac{\hat{\theta}}{\theta} - 1 \tag{8}$$

Under the entropy loss function (8), the Bayes estimator, denoted by $\hat{\theta}_{BE}$, can be derived as follows:

$$\hat{\theta}_{BE} = [E(\theta^{-1} \mid X)]^{-1} \tag{9}$$

Bayes estimation: In this Section, we will discuss Bayes estimation of the shape parameter θ under above loss functions. The results are reported in Theorem 1.

Theorem 1: Suppose that we observe n samples X_1, X_2, \dots, X_n from the Minimax distribution (1) and $x = (x_1, x_2, \dots, x_n)$ is the observation of $X = (X_1, X_2, \dots, X_n)$. Set $T = -\sum_{i=1}^n \ln(1 - X_i^c)$, the prior distribution of θ is quasi-prior (3), then:

i. Under squared error loss function $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, the Bayes estimator of θ , denoted by $\hat{\theta}_{BS}$, can be obtained as:

$$\hat{\theta}_{BS} = \frac{T}{n + d - 2}, \quad n + d > 2 \tag{10}$$

ii. The Bayes estimator $\hat{\theta}_{BL}$ under LINEX loss function (4), can be derived as:

$$\hat{\theta}_{BL} = \frac{T}{c} [1 - \exp(-\frac{c}{n + d})] \tag{11}$$

iii. The Bayes estimator $\hat{\theta}_{BP}$ under Precautionary loss function (6), can be obtained as:

$$\hat{\theta}_{BP} = \frac{T}{\sqrt{(n + d - 2)(n + d - 3)}} \tag{12}$$

iv. The Bayes estimator $\hat{\theta}_{BE}$ under entropy loss function (8), is given by:

$$\hat{\theta}_{BE} = \frac{T}{n + d - 1} \tag{13}$$

Proof: For given sample observation $x = (x_1, x_2, \dots, x_n)$, and, the likelihood function of θ is given by:

$$l(\theta; x) = \prod_{i=1}^n f(x_i; \theta, \alpha)$$

That is:

$$\begin{aligned} l(\theta; x) &= (\theta^{-1}\alpha)^n \left[\prod_{i=1}^n x_i \right]^{\alpha-1} \left[\prod_{i=1}^n (1-x_i^\alpha)^{\theta-1} \right] \\ &\propto \theta^{-n} \prod_{i=1}^n (1-x_i^\alpha)^{\theta-1} \\ &\propto \theta^{-n} e^{-\frac{t}{\theta}} \end{aligned} \tag{14}$$

where, $t = -\sum_{i=1}^n \ln(1-x_i^\alpha)$ is the observation of T .

Combining the quasi-prior distribution (3), with the likelihood function $l(\theta; x)$ using Bayes theorem, the posterior density of θ is given as:

$$\begin{aligned} \pi(\theta | x) &\propto l(\theta; x) \cdot \pi(\theta) \\ &\propto \theta^{-n} e^{-t/\theta} \cdot \theta^{-d} \propto \theta^{-(n+d)} e^{-t/\theta} \end{aligned}$$

It is obvious that $\theta | X$ is distributed with inverted Gamma distribution, denoted $I\Gamma(n+d-1, T)$. Then from the properties of inverted Gamma distribution, we have the following results:

$$E(\theta | X) = \frac{T}{n+d-2} \tag{15}$$

$$E(\theta^{-1} | X) = \frac{n+d-1}{T} \tag{16}$$

$$E(\theta^2 | X) = \frac{T^2}{(n+d-2)(n+d-3)} \tag{17}$$

Then, (i), (iii) and (iv) are directly proved. To see (ii), we need the following calculation:

$$\begin{aligned} &E\left[\frac{1}{\theta} \exp\left(-\frac{c\hat{\theta}_{BL}}{\theta}\right) | X\right] \\ &= \int_0^\infty \frac{1}{\theta} \exp\left(-\frac{c\hat{\theta}_{BL}}{\theta}\right) \frac{T^{n+d-1}}{\Gamma(n+d-1)} \theta^{-(n+d)} e^{-\frac{T}{\theta}} d\theta \\ &= \frac{T^{n+d-1}}{\Gamma(n+d-1)} \frac{\Gamma(n+d)}{(T-c\hat{\theta}_{BL})^{n+d}} \end{aligned}$$

By using Eq. (5) and (16), we can solve the Bayes estimator $\hat{\theta}_{BL}$ under LINEX loss function (4) and given by:

$$\hat{\theta}_{BL} = \frac{T}{c} \left[1 - \exp\left(-\frac{c}{n+d}\right) \right]$$

EMPIRICAL STUDY

To compare the Bayes estimators obtained in this study, Monte Carlo simulations are used to generate random samples of different sizes. The Mean Square Errors (MSEs) are used to compare the performance of different Bayes estimates. The simulations are generated as follows:

- Step 1:** For given $\alpha = 3$, $\theta = 0.5$, generate random samples of different sizes: $n = 10, 20, 30, 50, 75$ and 100 from the Minimax (α, θ) distribution
- Step 2:** Set $c = -1, -0.5, 0.5, 1$ and the Bayes estimates are computed according to the Eq. (10)-(13)

Table 1: Estimates and MSEs under different loss for different sample sizes with $d = 0$

n	$\hat{\theta}_{BL}$					$\hat{\theta}_{BP}$	$\hat{\theta}_{BE}$
	$\hat{\theta}_{BS}$	c = -1	c = -0.5	c = 0.5	c = 1.0		
10	0.6193 (0.0550)	0.5211 (0.0293)	0.5081 (0.0275)	0.4833 (0.0251)	0.4715 (0.0244)	0.6621 (0.0728)	0.5505 (0.0347)
20	0.5535 (0.0184)	0.5108 (0.0134)	0.5044 (0.0129)	0.4919 (0.0123)	0.4859 (0.0122)	0.5695 (0.0213)	0.5243 (0.0145)
30	0.5383 (0.0112)	0.5109 (0.0089)	0.5066 (0.0087)	0.4983 (0.0083)	0.4942 (0.0082)	0.5482 (0.0124)	0.5198 (0.0094)
50	0.5229 (0.0060)	0.5070 (0.0052)	0.5045 (0.0051)	0.4995 (0.0050)	0.4970 (0.0049)	0.5284 (0.0064)	0.5122 (0.0054)
75	0.5116 (0.0037)	0.5013 (0.0035)	0.4996 (0.0034)	0.4963 (0.0034)	0.4947 (0.0034)	0.5152 (0.0039)	0.5047 (0.0035)
100	0.5105 (0.0027)	0.5028 (0.0026)	0.5015 (0.0025)	0.4990 (0.0025)	0.4978 (0.0025)	0.5131 (0.0028)	0.5053 (0.0026)

Table 2: Estimates and MSEs under different loss for different sample sizes with $d = 1.0$

n	$\hat{\theta}_{BL}$					$\hat{\theta}_{BP}$	$\hat{\theta}_{BE}$
	$\hat{\theta}_{BS}$	c = -1	c = -0.5	c = 0.5	c = 1.0		
10	0.5505 (0.0347)	0.4715 (0.0244)	0.4608 (0.0241)	0.4403 (0.0242)	0.4403 (0.0245)	0.5839 (0.0433)	0.4955 (0.0261)
20	0.5243 (0.0145)	0.4859 (0.0122)	0.4801 (0.0121)	0.4688 (0.0121)	0.4688 (0.0122)	0.5387 (0.0162)	0.4981 (0.0126)
30	0.5198 (0.0094)	0.4942 (0.0082)	0.4902 (0.0082)	0.4823 (0.0081)	0.4823 (0.0081)	0.5290 (0.0102)	0.5024 (0.0085)
50	0.5122 (0.0054)	0.4970 (0.0049)	0.4946 (0.0049)	0.4897 (0.0049)	0.4897 (0.0049)	0.5175 (0.0056)	0.5020 (0.0050)
75	0.5047 (0.0035)	0.4947 (0.0034)	0.4930 (0.0034)	0.4898 (0.0034)	0.4898 (0.0034)	0.5081 (0.0036)	0.4980 (0.0034)
100	0.5053 (0.0026)	0.4978 (0.0025)	0.4965 (0.0025)	0.4941 (0.0025)	0.4941 (0.0025)	0.5079 (0.0027)	0.5003 (0.0025)

Step 3: The above steps are repeated $N = 2000$ times and the Mean Square Errors (MSEs) are computed for different sample sizes n . The Mean Square Error (MSE) is defined as:

$$ER(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$$

where $\hat{\theta}_i$ is the estimate at the i^{th} run.

The computational results are computed by MATLAB 19.0. The Bayes estimates and MSEs (in brackets) are reported in Table 1 and 2.

Numerical simulations expressed in Table 1 and 2 show that though under small sample sizes, the value of MSE under squared-error loss is larger than that of other Bayes estimates under asymmetric loss functions, but when the sample sizes are large (i.e., $n > 50$), the MSEs are almost equal, thus any of these Bayes estimators can be selected for the practical application.

CONCLUSION

In this study, we have derived Bayes estimators of the shape parameter of Minimax distribution under squared error loss, LINEX loss, Precautionary loss and entropy loss functions. Simulation results show that the Bayes estimates under the squared error loss function have the largest MSEs as compared with estimates under other loss functions. It is immediate to note that MSEs decrease as sample size increases. When the sample size n is large, such as $n > 50$, the MSEs are almost the same. The result concluded in this study can also be extended to other distributions, such as Poisson distribution, exponential and Rayleigh distribution.

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