

Research Article

Minimax Estimation of Parameter of Generalized Exponential Distribution under Square Log Error Loss and MLINEX Loss Functions

Lanping Li

Department of Basic Subjects, Hunan University of Finance and Economics,
Changsha 410205, P.R. China

Abstract: The aim of this paper is to study the minimax estimation of the generalized exponential distribution. Bayesian estimators and minimax estimators of the parameter of the generalized exponential distribution have been obtained under the well known weighted square error loss, square log error loss and Modified Linear Exponential (MLINEX) loss functions. Finally, Monte Carlo simulation is used to compare the efficiency of these minimax estimators by using mean square error.

Keywords: Bayes estimator, generalized exponential distribution, MLINEX loss, squared log error loss

INTRODUCTION

Generalized exponential distribution has been used as an alternative to gamma and Weibull distributions in many situations by Gupta and Kundu (1999). In recent years, an impressive array of papers has been devoted to study the behavioral patterns of the parameters of the generalized exponential distribution using both classical and Bayesian approach, a very good summary of this work can be found in Gupta and Kundu (1999, 2008), Raqab and Madi (2005) and Singh *et al.* (2008) and the references cited there for some recent developments on GE distribution.

Suppose X be a random variable from generalized exponential distribution (GE (θ)) if the Probability Density Function (PDF) is given by:

$$f(x; \theta) = \theta e^{-x} (1 - e^{-x})^{\theta-1}, \quad x > 0, \theta > 0 \quad (1)$$

where, θ is the shape parameter.

The minimax estimation is an upgraded non-classical approach in the estimation area of statistical inference, which has drawn great attention by many researchers. Podder *et al.* (2004) studied the minimax estimator of the parameter of the Pareto distribution under Quadratic and MLINEX loss functions. Dey (2008) studied the minimax estimator of the parameter of Rayleigh distribution. Shadrokh and Pazira (2010) studied the minimax estimator of the parameter of the minimax distribution under several loss functions.

In this study, weighted square error loss, log error squared error and MLINEX loss functions have been used to obtain the minimax estimators of the parameter of the GE (θ) distribution.

PRELIMINARIES

Let X_1, X_2, \dots, X_n be a sample from the generalized exponential distribution (1) and $x = (x_1, x_2, \dots, x_n)$ is the observation of $X = (X_1, X_2, \dots, X_n)$. The Likelihood Function (LF) of θ for the given sample observation is:

$$l(\theta; x) = \theta^n e^{-\sum_{i=1}^n x_i} \prod_{i=1}^n (1 - e^{-x_i})^{\theta-1} \quad (2)$$

That is:

$$l(\theta; x) \propto \theta^n e^{-\left(\sum_{i=1}^n -\ln(1 - e^{-x_i})\right)\theta} \propto \theta^n e^{-t\theta} \quad (3)$$

where, $t = \sum_{i=1}^n -\ln(1 - e^{-x_i})$ is the observation of $T = \sum_{i=1}^n -\ln(1 - e^{-X_i})$.

The maximum likelihood estimator of θ is easily derived as:

$$\hat{\theta}_M = \frac{n}{T} \quad (4)$$

And we can also show that $T = \sum_{i=1}^n -\ln(1 - e^{-X_i})$ is distributed the Gamma distribution $\Gamma(n, \theta)$.

In the Bayes approach, we further assume that some prior knowledge about the parameter θ is available to the investigation from past experience with the exponential model. The prior knowledge can often be summarized in terms of the so-called prior densities on parameter space of θ . In the following discussion, we assume the following Jeffrey's non-informative prior density defined as:

$$\pi(\theta) \propto \frac{1}{\theta}, \quad \theta > 0 \tag{5}$$

In Bayesian estimation, the loss function plays an important role and the squared error loss as the most common symmetric loss function is widely used due to its great analysis properties. And the weighted Square Error Loss Function (SELF):

$$L(\theta, \delta) = \frac{(\delta - \theta)^2}{\theta^2} \tag{6}$$

which is a symmetrical loss function that assigns equal losses to overestimation and underestimation. However, in many practical problems, overestimation and underestimation will make different consequents.

The squared log error loss function: Thus using of the symmetric loss function may be inappropriate; Brown (1968) proposed a new loss function for scale parameter estimation. This loss that is called squared log error loss is:

$$L(\theta, \delta) = (\ln \delta - \ln \theta)^2 \tag{7}$$

which is balanced and $L(\theta, \delta) \rightarrow \infty$ as $\delta \rightarrow 0$ or ∞ . This loss is not always convex, it is convex for $\frac{\delta}{\theta} \leq e$ and concave otherwise, but its risk function has minimum w.r.t.:

$$\hat{\delta}_{SL} = \exp[E(\ln \theta | X)]$$

The MLINEX loss function: Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX function, which is suitable for situations where overestimation is more costly than underestimation.

When estimating a parameter θ by $\hat{\theta}$, the MLINEX is given by (Podder *et al.*, 2004):

$$L(\theta, \delta) = \omega \left[\left(\frac{\delta}{\theta} \right)^c - c \ln \left(\frac{\delta}{\theta} \right) - 1 \right]; \omega > 0, c \neq 0 \tag{8}$$

Remark 1: For any prior distribution of θ , under the MLINEX loss function (8), we can show that the Bayes estimator of θ , denoted by $\hat{\delta}_{ML}$, is given by:

$$\hat{\delta}_{ML} = [E(\theta^{-c} | X)]^{-1/c} \tag{9}$$

Bayes estimation: Combining (3) with non-informative prior (5) using Bayes theorem, the posterior pdf of θ is given by:

$$h(\theta | X) = \frac{T^n}{\Gamma(n)} \theta^{n-1} e^{-T\theta} \tag{10}$$

This is a Gamma distribution $\Gamma(n, T)$. Then:

- The Bayes estimator under the weighted square error loss function is given by:

$$\hat{\delta}_{BS} = \frac{E[\theta^{-1} | X]}{E[\theta^{-2} | X]} = \frac{n-2}{T} \tag{11}$$

- Using (10):

$$\begin{aligned} E(\ln \theta | X) &= \frac{T^n}{\Gamma(n)} \int_0^\infty \ln \theta \cdot \theta^{n-1} e^{-T\theta} d\theta \\ &= \frac{d}{dr} \ln \Gamma(n) - \ln T = \Psi(n) - \ln T \end{aligned}$$

where,

$$\begin{aligned} \Psi(n) &= \frac{d}{dn} \ln \Gamma(n) \\ &= \int_0^{+\infty} \frac{\ln y \cdot y^{n-1} e^{-y}}{\Gamma(n)} dy \end{aligned}$$

is a Digamma function.

Then the Bayes estimator under the squared log error loss function is come out to be:

$$\hat{\delta}_{SL} = \exp[E(\ln \theta | X)] = \frac{e^{\Psi(n)}}{T} \tag{12}$$

- Using (10), the Bayes estimator under the MLINEX loss function is obtained as:

$$\hat{\delta}_{MML} = [E(\theta^{-c} | X)]^{-1/c} = \frac{1}{T} \cdot \left[\frac{\Gamma(n-c)}{\Gamma(n)} \right]^{-1/c} \tag{13}$$

MINIMAX ESTIMATION

The derivation of minimax estimators depends primarily on a theorem due to Lehmann which can be stated as follows:

Lemma 1: (Lehmann's Theorem) (Brown, 1968) If $\tau = \{F_\theta; \theta \in \Theta\}$ be a family of distribution functions and D a class of estimators of θ . Suppose that $\delta^* \in D$ is a Bayes estimator against a prior distribution $\delta^*(\theta)$ on Θ and $R(\delta^*, \theta)$ equals constant on Θ ; then δ^* is a minimax estimator of θ .

Theorem 1: Let X_1, X_2, \dots, X_n be a random sample drawn from the density (1), then $\hat{\delta}_{BS} = \frac{n-2}{T}$ is the

minimax estimator of the parameter θ for the weighted square error loss (6).

Theorem 2: Let X_1, X_2, \dots, X_n be a random sample drawn from the density (1), then $\hat{\delta}_{SL} = \frac{e^{\Psi(n)}}{T}$ is the minimax estimator of the parameter θ for the squared log error loss (7).

Theorem 3: Let X_1, X_2, \dots, X_n be a random sample drawn from the density (1), then:

$$\hat{\delta}_{MML} = \frac{1}{T} \cdot \left[\frac{\Gamma(n)}{\Gamma(n-c)} \right]^{1/c} = \frac{K}{T}, \quad K = \left[\frac{\Gamma(n)}{\Gamma(n-c)} \right]^{1/c}$$

is the minimax estimator of the parameter θ for the MLINEX loss (8).

Proof: First we have to prove the theorem 1. To prove the theorem we shall use Lehmann's theorem, which has been stated before. For this, first we have to find the Bayes estimator δ of θ . Then if we can show that the risk of δ is constant, then the theorem 1 will be proved. The risk function of the estimator $\hat{\delta}_{BS} = \frac{n-2}{T}$ is:

$$R(\theta) = E \left[L \left(\theta, \frac{n-2}{T} \right) \right] = E \left[\left(\frac{(n-2)/T - \theta}{\theta} \right)^2 \right]$$

$$= \frac{1}{\theta^2} \left[(n-2)^2 E(T^{-2}) - 2\theta(n-2)E(T^{-1}) + \theta^2 \right]$$

Then, $R(\theta) = \frac{1}{n-1}$ which is a constant. So, according to the Lehmann's theorem it follows that, $\hat{\delta}_{BS} = \frac{n-2}{T}$ is the minimax estimator for the parameter θ of the generalized exponential distribution under the quadratic loss function (7).

Now we are going to prove the theorem 2. The risk function of the estimator:

$$\hat{\delta}_{SL} = \frac{e^{\Psi(n)}}{T}$$

is:

$$R(\theta) = E \left[L \left(\theta, \hat{\delta}_{SL} \right) \right] = E \left[\left(\ln \hat{\delta}_{SL} - \ln \theta \right)^2 \right]$$

$$= E \left[\ln^2 \hat{\delta}_{SL} \right] - 2 \ln \theta \cdot E \left[\ln \hat{\delta}_{SL} \right] + (\ln \theta)^2$$

From the conclusion $T \sim \Gamma(n, \theta)$, we have:

$$E(T^{-1}) = \frac{\theta}{n-1}, \quad E(T^{-2}) = \frac{\theta^2}{(n-1)(n-2)}$$

$$E(\ln T) = \Psi(n) - \ln \theta$$

Thus:

$$E[\ln \hat{\delta}_{SL}] = E(\Psi(n) - \ln T)$$

$$= \Psi(n) - (\Psi(n) - \ln \theta) = \ln \theta$$

$$E[\ln^2 \hat{\delta}_{SL}] = E(\Psi(n) - \ln T)^2$$

$$= \Psi^2(n) - 2\Psi(n)E(\ln T) + E[(\ln T)^2]$$

Using the fact:

$$\Psi'(n) = \int_0^\infty \frac{(\ln y)^2 y^{n-1} e^{-y}}{\Gamma(n)} dy - \int_0^\infty \frac{(\ln y)^2 y^{n-1} e^{-y}}{\Gamma(n)} \Psi(n) dy$$

$$= E[(\ln Y)^2] - \Psi^2(n)$$

where, $Y \sim \Gamma(n, 1)$. Then we can show that:

$$Y = T\theta \sim \Gamma(n, 1)$$

Thus,

$$\Psi^2(n) + \Psi'(n) = E[(\ln Y)^2] = E[(\ln T + \ln \theta)^2]$$

$$= E[(\ln T)^2] + 2 \ln \theta \cdot (\Psi(n) - \ln \theta) + (\ln \theta)^2$$

Then we get the fact:

$$E[(\ln T)^2] = \Psi^2(n) + \Psi'(n) - 2 \ln \theta \cdot \Psi(n) + (\ln \theta)^2$$

Therefore $E[\ln^2 \hat{\delta}_{SL}] = \Psi'(n) + (\ln \theta)^2$.

Then $R(\theta) = \Psi'(n)$, which is a constant. So, according to the Lehmann's theorem it follows that, $\hat{\delta}_{SL} = \frac{e^{\Psi(n)}}{T}$ is the minimax estimator for the parameter θ of the generalized exponential distribution under the squared log error loss function (8).

Now we give the proof of the case of Theorem 3. The risk function under the MLINEX loss function (9) is:

$$R(\theta) = E \left[L \left(\theta, \hat{\delta}_{MML} \right) \right] = w E \left[\left(\frac{\hat{\delta}_{MML}}{\theta} \right)^c - c \ln \frac{\hat{\delta}_{MML}}{\theta} - 1 \right]$$

$$= \omega \left[\frac{1}{\theta^c} E(\hat{\delta}_{MML}^c) - c E(\ln \hat{\delta}_{MML}) + c \ln \theta - 1 \right]$$

But:

$$E(\hat{\delta}_{MML}^c) = E(K/T)^c = K^c \cdot E(T^{-c}) = K^c \frac{\Gamma(n-c)}{\Gamma(n)} \theta^c = \theta^c$$

$$E(\ln \hat{\delta}_{MML}) = E(\ln \frac{K}{T}) = \ln K - E(\ln T)$$

$$= \ln K + \ln \theta - \Psi(n)$$

Table 1: Estimated value and corresponding ER ($\hat{\theta}$)

n	Criteria	$\hat{\delta}_{ML}$	$\hat{\delta}_{BS}$	$\hat{\delta}_{SL}$	$\hat{\delta}_{MMLC} = -1.0$	$\hat{\delta}_{MMLC} = 1.0$	$\hat{\delta}_{MMLC} = 2.0$
10	Estimated value	1.1158	0.8927	1.0605	1.1158	1.0042	0.9468
	ER ($\hat{\theta}$)	0.1701	0.1118	0.1452	0.1701	0.1269	0.1156
20	Estimated value	1.0522	0.9469	1.0260	1.0522	0.9995	0.9729
	ER ($\hat{\theta}$)	0.0638	0.0523	0.0587	0.0638	0.0551	0.0529
50	Estimated value	1.0221	0.9812	1.0119	1.0221	1.0017	0.9914
	ER ($\hat{\theta}$)	0.0222	0.0203	0.0214	0.0222	0.0208	0.0205
75	Estimated value	1.0153	0.9882	1.0086	1.0153	1.0018	0.9950
	ER ($\hat{\theta}$)	0.0142	0.0133	0.0138	0.0142	0.0136	0.0134
100	Estimated value	1.0103	0.9901	1.0052	1.0103	1.0002	0.9951
	ER ($\hat{\theta}$)	0.0104	0.0100	0.0102	0.0104	0.0101	0.0100

Then $R(\theta) = w[\ln K^{-c} + c\Psi(n)]$, which is a constant. So, according to the Lehmann's theorem it follows that, $\hat{\delta}_{MML} = \frac{1}{T} \cdot \left[\frac{\Gamma(n)}{\Gamma(n-c)} \right]^{1/c}$ is the minimax estimator for the parameter θ of the generalized exponential distribution under the MLINEX loss function (9).

EMPIRICAL STUDY

The risks under squared-error loss of the estimates are considered to compare the different estimators of the parameter θ of the generalized exponential distribution that are obtained by maximum likelihood and minimax methods for weighted square error loss, squared log error error loss and MLINEX loss functions:

- Based on the given value $\theta = 1.0$, a sample of size n is then generated from the density of the Generalized Exponential distribution (1), which is considered to be the informative sample
- The MLE and minimax estimators is calculated based on Section 3 and in loss function (6), $c = -1.0, 1.0, 2.0$
- Steps (1) to (2) are repeated $N = 5000$, times and the risks under squared error loss of the estimates are computed by using:

$$ER(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$$

where $\hat{\theta}_i$ is the estimate at the i^{th} run.

CONCLUSION

The estimated values of the parameter and ER of the estimators are computed by the Monte-Carlo Simulation method from the generalized exponential distribution (1) with $\theta = 1.0$. It is seen that for small sample sizes ($n < 50$), minimax estimators for MLINEX loss function when $c = 1.0$ appear to be better than the other minimax estimators. But for large sample sizes ($n > 50$), all the three estimators have approximately the same ER. The obtained results are demonstrated in the Table 1.

ACKNOWLEDGMENT

This study is partially supported by Foundation of Hunan Educational Committee, No. 12C0563.

REFERENCES

Brown, L.D., 1968. Inadmissibility of the usual estimators of scale parameters in problems with unknown location and scale parameters. Ann. Math. Stat., 39: 29-48.

Dey, S., 2008. Minimax estimation of the parameter of the rayleigh distribution under quadratic loss function. Data Sci. J., 7: 23-30.

Gupta, R.D. and D. Kundu, 1999. Generalized exponential distribution. Aust. NZ J. Stat., 41(2): 173-188.

Gupta, R.D. and D. Kundu, 2008. Generalized exponential distribution: Bayesian estimations. Comput. Stat. Data Anal., 4: 1873-1883.

Podder, C.K., M.K. Roy, K.J. Bhuiyan and A. Karim, 2004. Minimax estimation of the parameter of the Pareto distribution for quadratic and MLINEX loss functions. Pak. J. Stat., 20(1): 137-149.

Raqab, M.Z. and T. Madi, 2005. Bayesian inference for the generalized exponential distribution. J. Stat. Comput. Sim., 75(10): 841-852.

Shadrokh, A. and H. Pazira, 2010. Minimax estimation on the minimax distribution. Int. J. Stat. Syst., 5(2): 99-118.

Singh, R., S.K. Singh, V. Singh and G.P. Singh, 2008. Bayes estimator of generalized exponential parameters under linex loss function using Lindley's approximation. Data Sci. J., 7: 65-75.

Varian, H.R., 1975. A Bayesian Approach to Real Estate Assessment. In: Fienberg, S.E. and A. Zellner (Eds.), Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage. North Holland Publishing Co., Amsterdam, pp: 195-208.

Zellner, A., 1986. Bayesian estimation and prediction using asymmetric loss functions. J. Am. Stat. Assoc., 81: 446-451.