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Research Article

Bayes Estimation of Parameter of Exponential Distribution under a Bounded Loss Function

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Abstract: The aim of this study is to study the Bayes estimation of parameter of exponential distribution under a bounded loss function, named reflected gamma loss function, which proposed by Towhidi and Behboodian (1999). The inverse Gamma prior distribution is used as the prior distribution of the parameter of exponential distribution. Bayesian estimators are obtained under squared error loss and the reflected gamma loss functions. Minimum risk equivariant estimator of the parameter is also derived. Finally, a numerical simulation is used to compare the estimators obtained.

Keywords: Bayes estimator, bounded loss function, minimum risk equivariant estimator, squared error loss

INTRODUCTION

Exponential distribution plays an important role in lifetime data analysis. Many authors have developed inference procedures for exponential model. For example, Kulldorff (1961) devoted a large part of book to the estimation of the parameters of the exponential distribution based on completely or partially grouped data. Sarhan (2003) obtained the empirical Bayes estimators of exponential model. Janeen (2004) discussed the empirical Bayes estimators of the parameter of parameter of exponential distribution based on record values. To see more details, one can see Balakrishnan *et al*. (2005) and Al-Hemyari (2009) and references therein.

Suppose that X is a variable drawn from a exponential distribution with the Probability Density Function (PDF):

$$f(x \mid \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad , x > 0 \tag{1}$$

where, $\theta > 0$ is the scale parameter.

For a Bayesian analysis, loss function plays an important role in it and the squared error loss and LINEX loss functions used by most researchers. These function are unbounded and widely employed in decision theory due to its elegant mathematical properties, not its applicability to the representation of a true loss structure (Leon and Wu, 1992). Various examples illustrate that in many situations, unbounded loss can be unduly restrictive and suggest that instead we should consider the properties of estimators based on a bounded loss function. A bounded loss avoids the potential explosion of the expected loss. Moreover, the nature of many decision problems and practical arguments require the use of bounded loss functions, especially in financial problems. For more details see Berger (1985). To overcome the shortcoming of unbounded loss, several bounded loss functions are proposed by many authors, for example, Spring (1993) proposed a bounded loss function named reflected normal loss. Towhidi and Behboodian (1999) proposed reflected gamma loss which is also a bounded loss function. Wen and Levy (2001) proposed a bounded asymmetric loss function called BLINEX loss function. To see more about the discussion of bounded loss function, one can reference Bartholomew and Spiring (2002) and Kamińska (2010).

This study will discuss the Bayes estimation of the parameter of exponential distribution under the following reflected gamma loss (Towhidi and Behboodian, 1999):

$$L(\hat{\theta}, \theta) = k[1 - (\hat{\theta} / \theta)^{q^2} e^{-q^2(\hat{\theta}/\theta - 1)}] \tag{2}$$

where, $q>0$ is a shape parameter and $k>0$ is the maximum loss parameter. Note that we can write the loss (2) as a monotone function of the entropy loss function:

$$L_1(\hat{\theta}, \theta) = \frac{\hat{\theta}}{\theta} - \ln \frac{\hat{\theta}}{\theta} - 1$$

In the following way:

$$\begin{aligned} L(\hat{\theta}, \theta) &= k[1 - \exp(-q^2(\frac{\hat{\theta}}{\theta} - \ln \frac{\hat{\theta}}{\theta} - 1))] \\ &= k[1 - e^{-q^2 L_1(\hat{\theta}, \theta)}] \end{aligned}$$

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This can be approximated by $kq^2 L_1(\hat{\theta},\theta)$, for small values of q, which is multiple of the entropy loss.

Under the reflected gamma loss function, Meghnatisi and Nematollahi (2009) studied the admissibility and inadmissibility of usual and mixed estimators of two ordered Gamma scale parameters. In this study, we will discuss the admissibility and inadmissibility of parameter of exponential distribution under various conditions.

MAXIMUM LIKELIHOOD ESTIMATION

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from exponential distribution (1). $(x_1, x_2, \ldots, x_n)$ is the observe value of $(X_1, X_2, \ldots, X_n)$. Then the likelihood function based on $(x_1, x_2, \ldots, x_n)$, is given by:

$$L(\theta \mid x) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^{-n} \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i) \tag{3}$$

The natural logarithm of likelihood function is given by:

$$\ln L(\theta \mid x) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i \tag{4}$$

Upon differentiating (4) with respect to θ and equating each results to zero. The MLE of θ is given by $\hat{\delta}_{MLE} = \overline{X}$, where, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

MINIMUM RISK EQUIVARIANT ESTIMATION

Consider a random sample $X_1, X_2, \ldots, X_n$ from exponential distribution (1). Let G be a group of transformations in the form $G = \{g_c | g_c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n), c>0\}$. Then we can show that the reflected gamma loss (2) and the decision problem are invariant and G and the class of all scale-invariant estimator of θ is of the form $\delta(X) = \delta_0(X)/W(Z)$, where δ_0 is any scale invariant estimator, $X = (X_1, X_2, \ldots, X_n)$ and $Z = (Z_1, Z_2, \ldots, Z_n)$ with $Z_i = X_i/X_n$, $i = 1, 2, \ldots, n-1$, $Z_n = X_n/|X_n|$. The proof of this conclusion can be founded in Lehmann (1983).

Moreover the best scale-invariant estimator, which also called Minimum Risk Equivariant (MRE) estimator δ^* is given by $\delta^*(X) = \delta_0(X)/w^*(Z)$, where, $w^*(Z)$ is a function of Z which maximize:

$$g(w) = E_{\theta=1}[(\frac{\delta_0(X)}{w(z)})^{q^2} \exp(-q^2(\frac{\delta_0(X)}{w(z)} - 1) \mid Z = z]$$

Lemma 1: If $\delta_0(X)$ is a finite risk scale invariant estimator of θ and when $\theta = 1, \delta_0(X)$ is assumed have the Gamma distribution $\Gamma(\alpha,\beta)$, with Probability Density Function (PDF):

$$\pi(x \mid \alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad \alpha,\beta > 0, x > 0 \tag{5}$$

where, α and β are known parameters and here α and β are independent of X.

Then the MRE estimator of θ under reflected gamma loss (2) is given by:

$$\delta^*(x) = \frac{\beta}{\alpha}\delta_0(x) \tag{6}$$

Proof: When $\theta = 1$ and $\delta_0(X)$ has the Gamma distribution $\Gamma(\alpha,\beta)$, then $g(w)$ is also be written as:

$$\begin{aligned} g(w) &= \int_0^\infty (\frac{x}{w})^{q^2} e^{-q^2(x/w-1)} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha e^{q^2}}{\Gamma(\alpha) w^{q^2}} \frac{\Gamma(\alpha+q^2)}{(\beta+q^2/w)^{\alpha+q^2}} \\ &= c(\alpha,\beta,q^2) \frac{w^\alpha}{(\beta+q^2/w)^{\alpha+q^2}} \end{aligned}$$

And $c(\alpha,\beta,q^2)$ is a function of α,β,q^2. By the equation $g'(w) = 0$ and because of $g'(w) > 0$. Then we can easily show that $w^* = \alpha/\beta$ maximizes the function $g(w)$. Hence, the Lemma 1 is proved.

Remark 1: Let $X_1, X_2, \ldots, X_n$ be a random sample from exponential distribution (1), then under the reflected gamma loss (2), the estimator $\delta_0(X) = \sum_{i=1}^{n} X_i$ is an equivariant estimator which has Γ $(n, 1)$ distribution when $\theta = 1$ and it follows from Basu's theorem that $\delta_0(X)$ is independent of Z. Hence, the MRE estimator of θ is:

$$\delta^*(X) = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} \cdot$$

BAYES ESTIMATION

In this section, we consider the Bayes estimation of the scale parameter θ in exponential distribution (1), in which the complete sufficient statistics $\delta_0(X) = \sum_{i=1}^{n} X_i = n\overline{X}$ has the distribution $\Gamma(n,\theta^{-1})$.

Assume that the conjugate family of prior distributions for $\lambda = \theta^{-1}$ is the family of Gamma distribution $\Gamma(\alpha,\beta)$. Note that the limiting case

$\alpha, \beta \to 0$ gives the usual non-informative prior $\pi(\lambda) \propto \lambda^{-1}$. The posterior distribution of λ is $\Gamma(n+\alpha, \beta+\delta_0(X))$ and the Bayes estimate of θ is a function $\delta(X)$ which maximize the function:

$$E[(\delta\lambda)^{q^2} \exp(-q^2(\delta\lambda-1) \mid X=x]$$
$$= \int_0^\infty (\delta\lambda)^{q^2} e^{-q^2(\delta\lambda-1)} \cdot \frac{(\beta+\delta_0(x))^{n+\alpha}}{\Gamma(n+\alpha)} \cdot \lambda^{n+\alpha-1} e^{-(\beta+\delta_0(x))\lambda} d\lambda$$

$$= \frac{(\delta e)^{q^2} (\beta+\delta_0(x))^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \lambda^{n+\alpha+q^2-1} e^{-(\beta+\delta_0(x)+\delta q^2)\lambda} d\lambda$$

or the function:

$$g(\delta) = \frac{\delta^{q^2}}{(\beta+\delta_0(x)+\delta q^2)^{n+\alpha+q^2}}$$

Then the Bayes estimator of θ, denoted by $\hat{\delta}_B$ is given by the solution of the equation $\frac{dg(\delta)}{d\delta} = 0$

Hence, we can show that $\hat{\delta}_B$ is given by:

$$\hat{\delta}_B = \frac{\beta+\delta_0(X)}{n+\alpha} \tag{7}$$

Remark 2: For the non-informative prior $\pi(\lambda) \propto \lambda^{-1}$, the posterior distribution of λ is $\Gamma(n, \delta_0(X))$ and we obtain the generalized Bayes estimator:

$$\hat{\delta}_* = \frac{\delta_0(X)}{n} = \overline{X}$$

EMPIRICAL BAYES ESTIMATION

In the former discussion, the Bayes estimator in (7) is seen to depend on the parameter β. When the prior parameter β is unknown, we may use the empirical Bayes approach to get its estimate. From (3) and (5), we calculate the marginal PDF of X, with density:

$$m(x \mid \beta) = \int_0^\infty L(x \mid \theta)\pi(\theta \mid \beta) d\theta$$

$$= \int_0^\infty \theta^{-n} \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i) \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} \exp(-\frac{\beta}{\theta}) d\theta$$

$$= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(n+\alpha+1)} \exp(-\frac{\beta+\sum_{i=1}^n x_i}{\theta}) d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(n+\alpha)}{(\beta+\sum_{i=1}^n x_i)^{n+\alpha}}$$

Based on $m(x|\beta)$, we obtain an estimator, $\hat{\beta}$ of β. The MLE of β is $\hat{\beta} = \frac{\alpha}{n}\sum_{i=1}^n X_i = \alpha\overline{X}$.

Now, by substituting $\hat{\beta}$ for β in the Bayes estimator (8), we obtain the empirical Bayes estimator as:

$$\hat{\delta}_{EB} = \frac{\hat{\beta}+\delta_0(X)}{n+\alpha} = \overline{X}.$$

NUMERICAL SIMULATION

To compare the different estimators of the parameter θ of the exponential distribution, the risks under squared error loss of the estimates are considered. These estimators are obtained by maximum likelihood and Bayes methods under reflected gamma loss function:

(i) Based on the given value $\theta = 1.0$, a sample of size n is then generated from the density of the exponential distribution (1), which is considered to be the informative sample.
(ii) The MLE and Bayes estimators are calculated based on Section 2 and q =1.0.
(iii) Steps (i) to (ii) are repeated N = 2000 times and the risks under squared error loss of the estimates are computed by using:

$$ER(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^N (\hat{\theta}-\theta)^2$$

where, $\hat{\theta}_i$ is the estimate at the i^{th} run.

CONCLUSION

The estimated values of the parameter and *ER* of the estimators are computed by the Monte-Carlo Simulation from the exponential distribution (1) with θ = 1.0. It is seen that for small sample sizes (n<50), estimators under reflected gamma loss function have smaller *ER* when choosing proper parameters α and β. But for large sample sizes (n>50), all the estimators have approximately the same *ER*. The obtained results are demonstrated in the Table 1.

Table 1: Estimated value and corresponding ER ($\hat{\theta}$)

			Bayes estimate		
n	Criteria	MLE	$\alpha = 0, \beta = 0.5$	$\alpha = 0.5, \beta = 1$	$\alpha = 1.0, \beta = 1.5$
10	Estimated value	0.9992	1.0492	1.0469	1.0447
	ER ($\hat{\theta}$)	0.1031	0.1056	0.0957	0.0872
25	Estimated value	1.0023	1.0223	1.0218	1.0214
	ER ($\hat{\theta}$)	0.0405	0.0410	0.0394	0.0379
50	Estimated value	0.9998	1.0098	1.0097	1.0096
	ER ($\hat{\theta}$)	0.0193	0.0194	0.0190	0.0186
75	Estimated value	0.9974	1.0041	1.0041	1.0040
	ER ($\hat{\theta}$)	0.0135	0.0135	0.0133	0.0131
100	Estimated value	1.0020	1.0070	1.0070	1.0069
	ER ($\hat{\theta}$)	0.0099	0.0099	0.0098	0.0097
125	Estimated value	0.9962	1.0002	1.0002	1.0002
	ER ($\hat{\theta}$)	0.0082	0.0081	0.0081	0.0080
150	Estimated value	0.9974	1.0007	1.0007	1.0007
	ER ($\hat{\theta}$)	0.0068	0.0068	0.0067	0.0067

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REFERENCES

Al-Hemyari, Z.A., 2009. Reliability function estimator with exponential failure model for engineering data. Proceeding of the World Congress on Engineering 2009, London, U.K.

Balakrishnan, N., C.T. Lin and P.S. Chan, 2005. A comparison of two simple prediction intervals for exponential distribution. IEEE T. Reliab., 54(1): 27-33.

Bartholomew, L. and F.A. Spiring, 2002. The inverted beta loss function: Properties and application. Transactions, 34: 1101-1109.

Berger, J.O., 1985. Statistical Decision Theory and Bayesian Analysis. 2nd Edn., Spring-Verlag, New York.

Janeen, Z.F., 2004. Empirical Bayes analysis of record statistics based on LINEX and quadratic loss functions. Comput. Math. Appl., 47: 947-954.

Kamińska, A., 2010. The equivalence of Bayes and robust Bayes estimators for various loss functions. Stat. Papers, 51: 179-191.

Kulldorff, G., 1961. Contributions to the Theory of Estimation from Grouped and Partially Grouped Samples. Almqvist and Wiksell, Stockholm, pp: 144.

Lehmann, E.L., 1983. Theory of Point Estimation. John and Wiley, New York, pp: 175-179.

Leon, R.W. and C.F. Wu, 1992. A theory of performance measures in parameter design. Stat. Sin., 2: 335-358.

Meghnatisi, Z. and N. Nematollahi, 2009. Inadmissibility of usual and mixed estimators of two ordered gamma scale parameters under reflected gamma loss function. J. Math. Extension, 3(2): 89-99.

Sarhan, A.A., 2003. Empirical Bayes estimetes in exponential reliability model. Appl. Math. Comput., 135: 319-332.

Spring, F.A., 1993. The reflected normal loss function. Can. J. Stat., 21(3): 321-330.

Towhidi, M. and J. Behboodian, 1999. Estimation of scale parameter under a reflected gamma loss function. J. Sci. I. R. Iran, 10(4): 265-269.

Wen, D. and M.S. Levy, 2001. Admissibility of Bayes estimates under BLINEX loss for normal mean problem. Commun. Stat. Theor. Methods, 30: 155-163.