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Research Article

A Modified Block Adam Moulton (MOBAM) Method for the Solution of Stiff Initial Value Problems of Ordinary Differential Equations

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Abstract: Stiff ordinary differential equations pose computational difficulties as they present severe step size restrictions on the numerical methods to be used. Construction of numerical methods that possess suitable stability properties for the solution of such systems has been the target of many researchers. Development of methods suitable for these systems of equations has been either through the use of derivative of the solution or by introducing off-step points, additional stages or super future points. These processes have been exploited in Runge-Kutta methods or linear multistep methods. In this study, an improved class of linear multistep block method has been constructed based on Adams Moulton block methods. The improved methods are shown to be A-stable, a property desirable to handle stiff ODEs. Methods of uniform orders 10 and 11 have been constructed. The efficiency of the new methods tested on stiff systems of ODEs and the results reveal that the MOBAM methods compare favourably with results obtained using the state of the art Matlab Ode23 solver.

Keywords: Block method, initial value problems, non-linear, stability

INTRODUCTION

Stiff initial value problems of ordinary differential equations were realized in 1952 arising from the study of chemical kinetics. Since then, many researches have been involved in the study and development of suitable numerical methods to handle them. Curtiss and Hirschfelder (1952), Mitchell and Craggs (1953) and Gear (1971) developed the Backward Differential Formulae which were used to solve stiff problems arising from chemical kinetics. Intensive work done in this regard yielding several efficient numerical algorithms has appeared in the literature. Because of the severe restriction on step size placed on Adams Moulton methods for the solution of stiff ODEs, Garfinket *et al*. (1978) introduced the concept of A-stability for linear multistep methods. This property became the minimum requirement for any linear multistep method to be used for the solution of stiff ODEs. Achieving A-stability was difficult, other stability requirements such as A(α)-stability and stiffly stability were considered. Chakrararti and Kamel (1983) developed stiffly stable second derivative methods with high order and improved stability regions to handle stiff problems. Several other researchers such as Samir (2013), Evelyn and Renante (2006), Song (2010), Vlachos *et al*. (2009) and Ali and Gholamreza (2012) developed improved and more friendly numerical methods with improved stability properties that could handle stiff ODEs with various degrees of stiffness.

In this study, we pursue the path of Adams Moulton methods by constructing A-stable block linear multistep methods of uniform order 10 and 11. These methods are obtained by modifying the Adams Moulton method of step number 9 by reducing its interpolation step number from 8 to 3. This approach is similar to that of Brugnano and Trigiante (1998) where they developed the Generalized Adams methods.

The modified methods are constructed using the concept of multistep collocation of Lie and Norsett (1989) which Onumanyi *et al*. (1994, 1999) referred to as block linear multistep methods. The block methods are arrived at by evaluating the continuous formulation of the new method at grid and off grid points to yield the discrete schemes used as block integrators. The methods are applied simultaneously producing self-starting methods thus eliminating the issue of overlap of pieces of solutions.

CONSTRUCTION OF THE NEW METHOD

The k-step general linear multistep method for numerically solving the differential equation:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a,b], \quad y \in R \quad (1)$$

is described by the linear difference equation:

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (2)$$

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With h being the step size, $\alpha_j(x)$ and $\beta_j(x)$ being the continuous coefficients of the method.
The new improved nine step method based on the Adams Moulton method is defined as:

$$y_{n+v} - \alpha_{v-1}(x) y_{n+v-1} = h \sum_{j=0}^{m} \beta_j(x) f_{n+j} \tag{3}$$

where, $v = \frac{k-1}{2}, \alpha_{v-1}$ and $\beta_j(x)$ are the continuous coefficients of the method, m is the number of distinct collocation points, h is the step size and $k = m - 1 = 3,5,7,9,...$

Using the method in Onumanyi *et al*. (1994, 1999) and further studied by Kumleng *et al*. (2012) and Chollom *et al*. (2012) the method (3) for $v = \frac{k-1}{2}$ and j =0, 1… 9 sequence in the matrix form:

$$DC = I \tag{4}$$

where, $D = C^{-1}$, C^{-1} being the elements of the continuous coefficients of the method.

Construction of MOBAM K = 9: This method has its general form as:

$$y(x) = \alpha_3(x) y_{n+3} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j}, \; j = 0,1,...,9 \tag{5}$$

Using the procedure in Onumanyi *et al*. (1999), we obtain the D matrix for (5) as:

$$D = \begin{bmatrix} 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 & 10x_{n+1}^9 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 & 10x_{n+2}^9 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 & 10x_{n+3}^9 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 & 10x_{n+4}^9 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & 9x_{n+5}^8 & 10x_{n+5}^9 \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 & 8x_{n+6}^7 & 9x_{n+6}^8 & 10x_{n+6}^9 \\ 0 & 1 & 2x_{n+7} & 3x_{n+7}^2 & 4x_{n+7}^3 & 5x_{n+7}^4 & 6x_{n+7}^5 & 7x_{n+7}^6 & 8x_{n+7}^7 & 9x_{n+7}^8 & 10x_{n+7}^9 \\ 0 & 1 & 2x_{n+8} & 3x_{n+8}^2 & 4x_{n+8}^3 & 5x_{n+8}^4 & 6x_{n+8}^5 & 7x_{n+8}^6 & 8x_{n+8}^7 & 9x_{n+8}^8 & 10x_{n+8}^9 \\ 0 & 1 & 2x_{n+9} & 3x_{n+9}^2 & 4x_{n+9}^3 & 5x_{n+9}^4 & 6x_{n+9}^5 & 7x_{n+9}^6 & 8x_{n+9}^7 & 9x_{n+9}^8 & 10x_{n+9}^9 \end{bmatrix} \tag{6}$$

Using the Maple software, the inverse of the matrix (6) is obtained and represented as C, the elements of which yield the continuous coefficients $\alpha_3(x)$ and $\beta_j(x)$, $j = 0,1,…9$ of the method (5) as:

$$\begin{aligned}
&\alpha_3(x) = 1 \\
&\beta_0(x) = (-\tfrac{3591h}{12800} + \mu - \tfrac{7129\mu^2}{5040h} + \tfrac{6515\mu^3}{6048h^2} - \tfrac{4523\mu^4}{9072h^3} + \tfrac{19\mu^5}{128h^4} - \tfrac{3013\mu^6}{103680h^5} + \tfrac{5\mu^7}{1344h^6} - \tfrac{29\mu^8}{96768h^7} + \tfrac{\mu^9}{72576h^8} - \tfrac{\mu^{10}}{3628800h^9}) \\
&\beta_1(x) = (-\tfrac{147429h}{89600} + \tfrac{9\mu^2}{2h} - \tfrac{4609\mu^3}{840h^2} + \tfrac{14139\mu^4}{4480h^3} - \tfrac{7667\mu^5}{7200h^4} + \tfrac{7807\mu^6}{34560h^5} - \tfrac{11\mu^7}{360h^6} + \tfrac{59\mu^8}{23040h^7} - \tfrac{11\mu^9}{90720h^8} + \tfrac{\mu^{10}}{403200h^9}) \\
&\beta_2(x) = (\tfrac{279h}{1600} - \tfrac{9\mu^2}{h} + \tfrac{5869\mu^3}{420h^2} - \tfrac{20837\mu^4}{2240h^3} + \tfrac{24901\mu^5}{7200h^4} - \tfrac{6787\mu^6}{8640h^5} + \tfrac{563\mu^7}{5040h^6} - \tfrac{7\mu^8}{720h^7} + \tfrac{43\mu^9}{90720h^8} - \tfrac{\mu^{10}}{100800h^9}) \\
&\beta_3(x) = (-\tfrac{13273h}{5600} + \tfrac{14\mu^2}{h} - \tfrac{6289\mu^3}{270h^2} + \tfrac{72569\mu^4}{4320h^3} - \tfrac{4013\mu^5}{600h^4} + \tfrac{13873\mu^6}{8640h^5} - \tfrac{401\mu^7}{1680h^6} + \tfrac{31\mu^8}{1440h^7} - \tfrac{7\mu^9}{6480h^8} + \tfrac{\mu^{10}}{43200h^9}) \\
&\beta_4(x) = (\tfrac{19107h}{8960} - \tfrac{63\mu^2}{4h} + \tfrac{6499\mu^3}{240h^2} - \tfrac{6519\mu^4}{320h^3} + \tfrac{122249\mu^5}{14400h^4} - \tfrac{36769\mu^6}{17280h^5} + \tfrac{3313\mu^7}{10080h^6} - \tfrac{353\mu^8}{11520h^7} + \tfrac{41\mu^9}{25920h^8} - \tfrac{\mu^{10}}{28800h^9}) \\
&\beta_5(x) = (-\tfrac{73809h}{44800} + \tfrac{63\mu^2}{5h} - \tfrac{265\mu^3}{12h^2} + \tfrac{1091\mu^4}{64h^3} - \tfrac{5273\mu^5}{720h^4} + \tfrac{32773\mu^6}{17280h^5} - \tfrac{305\mu^7}{1008h^6} + \tfrac{67\mu^8}{2304h^7} - \tfrac{\mu^9}{648h^8} + \tfrac{\mu^{10}}{28800h^9}) \\
&\beta_6(x) = (\tfrac{5039h}{5600} - \tfrac{7\mu^2}{h} + \tfrac{6709\mu^3}{540h^2} - \tfrac{84307\mu^4}{8640h^3} + \tfrac{10279\mu^5}{2400h^4} - \tfrac{9823\mu^6}{8640h^5} + \tfrac{313\mu^7}{1680h^6} - \tfrac{53\mu^8}{2880h^7} + \tfrac{13\mu^9}{12960h^8} - \tfrac{\mu^{10}}{43200h^9}) \\
&\beta_7(x) = (-\tfrac{3663h}{11200} + \tfrac{18\mu^2}{7h} - \tfrac{967\mu^3}{210h^2} + \tfrac{4101\mu^4}{1120h^3} - \tfrac{2939\mu^5}{1800h^4} + \tfrac{3817\mu^6}{8640h^5} - \tfrac{373\mu^7}{5040h^6} + \tfrac{151\mu^8}{20160h^7} - \tfrac{19\mu^9}{45360h^8} + \tfrac{\mu^{10}}{100800h^9}) \\
&\beta_8(x) = (\tfrac{909h}{12800} - \tfrac{9\mu^2}{16h} + \tfrac{3407\mu^3}{33360h^2} - \tfrac{1823\mu^4}{2240h^3} + \tfrac{10579\mu^5}{28800h^4} - \tfrac{3487\mu^6}{34560h^5} + \tfrac{347\mu^7}{20160h^6} - \tfrac{41\mu^8}{23040h^7} + \tfrac{37\mu^9}{362880h^8} - \tfrac{\mu^{10}}{403200h^9}) \\
&\beta_9(x) = (-\tfrac{25h}{3584} + \tfrac{\mu^2}{18h} - \tfrac{761\mu^3}{7560h^2} + \tfrac{29531\mu^4}{362880h^3} - \tfrac{89\mu^5}{2400h^4} + \tfrac{1069\mu^6}{103680h^5} - \tfrac{\mu^7}{560h^6} + \tfrac{13\mu^8}{69120h^7} - \tfrac{\mu^9}{90720h^8} + \tfrac{\mu^{10}}{3628800h^9})
\end{aligned} \tag{7}$$

Substituting the continuous coefficients (7) into (5) yields the continuous form of the MOBAM method (8):

$$
\begin{aligned}
y(x_n+\mu) = y_{n+3} &+ \left(-\frac{3591h}{12800}+\mu-\frac{7129\mu^2}{5040h}+\frac{6515\mu^3}{6048h^2}-\frac{4523\mu^4}{9072h^3}+\frac{19\mu^5}{128h^4}-\frac{3013\mu^6}{103680h^5}+\frac{5\mu^7}{1344h^6}-\frac{29\mu^8}{96768h^7}+\frac{\mu^9}{72576h^8}-\frac{\mu^{10}}{3628800h^9}\right)f_n + \\
&\left(-\frac{147429h}{89600}+\frac{9\mu^2}{2h}-\frac{4609\mu^3}{840h^2}+\frac{14139\mu^4}{4480h^3}-\frac{7667\mu^5}{7200h^4}+\frac{7807\mu^6}{34560h^5}-\frac{11\mu^7}{360h^6}+\frac{59\mu^8}{23040h^7}-\frac{11\mu^9}{90720h^8}+\frac{\mu^{10}}{403200h^9}\right)f_{n+1} + \\
&\left(\frac{279h}{1600}-\frac{9\mu^2}{h}+\frac{5869\mu^3}{420h^2}-\frac{20837\mu^4}{2240h^3}+\frac{24901\mu^5}{7200h^4}-\frac{6787\mu^6}{8640h^5}+\frac{563\mu^7}{5040h^6}-\frac{7\mu^8}{720h^7}+\frac{43\mu^9}{90720h^8}-\frac{\mu^{10}}{100800h^9}\right)f_{n+2} + \\
&\left(-\frac{13273h}{5600}+\frac{14\mu^2}{h}-\frac{6289\mu^3}{270h^2}+\frac{72569\mu^4}{4320h^3}-\frac{4013\mu^5}{600h^4}+\frac{13873\mu^6}{8640h^5}-\frac{401\mu^7}{1680h^6}+\frac{31\mu^8}{1440h^7}-\frac{7\mu^9}{6480h^8}+\frac{\mu^{10}}{43200h^9}\right)f_{n+3} + \\
&\left(\frac{19107h}{8960}-\frac{63\mu^2}{4h}+\frac{6499\mu^3}{240h^2}-\frac{6519\mu^4}{320h^3}+\frac{122249\mu^5}{14400h^4}-\frac{36769\mu^6}{17280h^5}+\frac{3313\mu^7}{10080h^6}-\frac{353\mu^8}{11520h^7}+\frac{41\mu^9}{25920h^8}-\frac{\mu^{10}}{28800h^9}\right)f_{n+4} + \\
&\left(-\frac{73809h}{44800}+\frac{63\mu^2}{5h}-\frac{265\mu^3}{12h^2}+\frac{1091\mu^4}{64h^3}-\frac{5273\mu^5}{720h^4}+\frac{32773\mu^6}{17280h^5}-\frac{305\mu^7}{1008h^6}+\frac{67\mu^8}{2304h^7}-\frac{\mu^9}{648h^8}+\frac{\mu^{10}}{28800h^9}\right)f_{n+5} + \\
&\left(\frac{5039h}{5600}-\frac{7\mu^2}{h}+\frac{6709\mu^3}{540h^2}-\frac{84307\mu^4}{8640h^3}+\frac{10279\mu^5}{2400h^4}-\frac{9823\mu^6}{8640h^5}+\frac{313\mu^7}{1680h^6}-\frac{53\mu^8}{2880h^7}+\frac{13\mu^9}{12960h^8}-\frac{\mu^{10}}{43200h^9}\right)f_{n+6} + \\
&\left(-\frac{3663h}{11200}+\frac{18\mu^2}{7h}-\frac{967\mu^3}{210h^2}+\frac{4101\mu^4}{1120h^3}-\frac{2939\mu^5}{1800h^4}+\frac{3817\mu^6}{8640h^5}-\frac{373\mu^7}{5040h^6}+\frac{151\mu^8}{20160h^7}-\frac{19\mu^9}{45360h^8}+\frac{\mu^{10}}{100800h^9}\right)f_{n+7} + \\
&\left(\frac{909h}{12800}-\frac{9\mu^2}{16h}+\frac{3407\mu^3}{33360h^2}-\frac{1823\mu^4}{2240h^3}+\frac{10579\mu^5}{28800h^4}-\frac{3487\mu^6}{34560h^5}+\frac{347\mu^7}{20160h^6}-\frac{41\mu^8}{23040h^7}+\frac{37\mu^9}{362880h^8}-\frac{\mu^{10}}{403200h^9}\right)f_{n+8} + \\
&\left(-\frac{25h}{3584}+\frac{\mu^2}{18h}-\frac{761\mu^3}{7560h^2}+\frac{29531\mu^4}{362880h^3}-\frac{89\mu^5}{2400h^4}+\frac{1069\mu^6}{103680h^5}-\frac{\mu^7}{560h^6}+\frac{13\mu^8}{69120h^7}-\frac{\mu^9}{90720h^8}+\frac{\mu^{10}}{3628800h^9}\right)f_{n+9}
\end{aligned}
\tag{8}
$$

where, $\mu = x - x_n$ and $\mu \in [0,\ 9h]$. Evaluating the continuous scheme (8) at the following points $\mu = 0, h, 2h, 4h, 5h, 6h, 7h, 8h, 9h$ yields the MOBAM method (9) for k = 9 used in block for the solution of ODEs:

$$
\begin{aligned}
y_{n+3} - y_n &= \tfrac{h}{89600}(25137 f_n + 147429 f_{n+1} - 15624 f_{n+2} + 212368 f_{n+3} - 191070 f_{n+4} + \\
&\quad 147618 f_{n+5} - 80624 f_{n+6} + 29304 f_{n+7} - 6363 f_{n+8} + 625 f_{n+9}) \\
y_{n+1} - y_{n+3} &= \tfrac{h}{113400}(729 f_n - 38938 f_{n+1} - 156340 f_{n+2} - 18742 f_{n+3} - 28234 f_{n+4} + \\
&\quad 24266 f_{n+5} - 13492 f_{n+6} + 4910 f_{n+7} - 1063 f_{n+8} + 104 f_{n+9}) \\
y_{n+2} - y_{n+3} &= \tfrac{h}{7257600}(-10625 f_n + 163531 f_{n+1} - 3133688 f_{n+2} - 5597072 f_{n+3} + 2166334 f_{n+4} \\
&\quad - 1295810 f_{n+5} + 617584 f_{n+6} - 206072 f_{n+7} + 42187 f_{n+8} - 3969 f_{n+9}) \\
y_{n+4} - y_{n+3} &= \tfrac{h}{7257600}(-3969 f_n + 50315 f_{n+1} - 342136 f_{n+2} + 3609968 f_{n+3} + 4763582 f_{n+4} - \\
&\quad 1166146 f_{n+5} + 462320 f_{n+6} - 141304 f_{n+7} + 27467 f_{n+8} - 2497 f_{n+9}) \\
y_{n+5} - y_{n+3} &= \tfrac{h}{113400}(-23 f_n + 334 f_{n+1} - 2804 f_{n+2} + 46378 f_{n+3} + 139030 f_{n+4} + 46378 f_{n+5} - \\
&\quad 2804 f_{n+6} + 334 f_{n+7} - 23 f_{n+8}) \\
y_{n+6} - y_{n+3} &= \tfrac{h}{89600}(-49 f_n + 603 f_{n+1} - 3960 f_{n+2} + 42352 f_{n+3} + 95454 f_{n+4} + 95454 f_{n+5} + \\
&\quad 42352 f_{n+6} - 3960 f_{n+7} + 603 f_{n+8} - 49 f_{n+9}) \\
y_{n+7} - y_{n+3} &= \tfrac{h}{14175}(13 f_{n+1} - 224 f_{n+2} + 5494 f_{n+3} + 17632 f_{n+4} + 10870 f_{n+5} + 17632 f_{n+6} + \\
&\quad 5494 f_{n+7} - 224 f_{n+8} + 13 f_{n+9}) \\
y_{n+8} - y_{n+3} &= \tfrac{h}{290304}(-425 f_n + 4675 f_{n+1} - 25400 f_{n+2} + 171760 f_{n+3} + 247150 f_{n+4} + \\
&\quad 381550 f_{n+5} + 185200 f_{n+6} + 387400 f_{n+7} + 101635 f_{n+8} - 2025 f_{n+9}) \\
y_{n+9} - y_{n+3} &= \tfrac{h}{1400}(9 f_n - 90 f_{n+1} + 396 f_{n+2} - 598 f_{n+3} + 3798 f_{n+4} - 1494 f_{n+5} + \\
&\quad 3980 f_{n+6} - 306 f_{n+7} + 2313 f_{n+8} + 392 f_{n+9})
\end{aligned}
\tag{9}
$$

Construction of Hybrid MOBAM K = 9, $\mu = \frac{17}{2}$: This is the hybrid form of the MOBAM family for k=9. An off step point is inserted in (5) to give its general form as:

$$y(x) = \alpha_3(x) y_{n+3} + h\sum_{j=0}^{k} \beta_j(x) f_{n+j} + h\beta_u(x) f_{n+u},\ j = 0,1,...,9,\ \mu = \tfrac{17}{2} \tag{10}$$

Following a similar procedure as in above section gives the matrix:

$$D=\begin{bmatrix}
1 & x_{n+3} & x_{n+3}^{2} & x_{n+3}^{3} & x_{n+3}^{4} & x_{n+3}^{5} & x_{n+3}^{6} & x_{n+3}^{7} & x_{n+3}^{8} & x_{n+3}^{9} & x_{n+3}^{10} & x_{n+3}^{11} \\
0 & 1 & 2x_{n} & 3x_{n}^{2} & 4x_{n}^{3} & 5x_{n}^{4} & 6x_{n}^{5} & 7x_{n}^{6} & 8x_{n}^{7} & 9x_{n}^{8} & 10x_{n}^{9} & 11x_{n}^{10} \\
0 & 1 & 2x_{n+1} & 3x_{n+1}^{2} & 4x_{n+1}^{3} & 5x_{n+1}^{4} & 6x_{n+1}^{5} & 7x_{n+1}^{6} & 8x_{n+1}^{7} & 9x_{n+1}^{8} & 10x_{n+1}^{9} & 11x_{n+1}^{10} \\
0 & 1 & 2x_{n+2} & 3x_{n+2}^{2} & 4x_{n+2}^{3} & 5x_{n+2}^{4} & 6x_{n+2}^{5} & 7x_{n+2}^{6} & 8x_{n+2}^{7} & 9x_{n+2}^{8} & 10x_{n+2}^{9} & 11x_{n+2}^{10} \\
0 & 1 & 2x_{n+3} & 3x_{n+3}^{2} & 4x_{n+3}^{3} & 5x_{n+3}^{4} & 6x_{n+3}^{5} & 7x_{n+3}^{6} & 8x_{n+3}^{7} & 9x_{n+3}^{8} & 10x_{n+3}^{9} & 11x_{n+3}^{10} \\
0 & 1 & 2x_{n+4} & 3x_{n+4}^{2} & 4x_{n+4}^{3} & 5x_{n+4}^{4} & 6x_{n+4}^{5} & 7x_{n+4}^{6} & 8x_{n+4}^{7} & 9x_{n+4}^{8} & 10x_{n+4}^{9} & 11x_{n+4}^{10} \\
0 & 1 & 2x_{n+5} & 3x_{n+5}^{2} & 4x_{n+5}^{3} & 5x_{n+5}^{4} & 6x_{n+5}^{5} & 7x_{n+5}^{6} & 8x_{n+5}^{7} & 9x_{n+5}^{8} & 10x_{n+5}^{9} & 11x_{n+5}^{10} \\
0 & 1 & 2x_{n+6} & 3x_{n+6}^{2} & 4x_{n+6}^{3} & 5x_{n+6}^{4} & 6x_{n+6}^{5} & 7x_{n+6}^{6} & 8x_{n+6}^{7} & 9x_{n+6}^{8} & 10x_{n+6}^{9} & 11x_{n+6}^{10} \\
0 & 1 & 2x_{n+7} & 3x_{n+7}^{2} & 4x_{n+7}^{3} & 5x_{n+7}^{4} & 6x_{n+7}^{5} & 7x_{n+7}^{6} & 8x_{n+7}^{7} & 9x_{n+7}^{8} & 10x_{n+7}^{9} & 11x_{n+7}^{10} \\
0 & 1 & 2x_{n+8} & 3x_{n+8}^{2} & 4x_{n+8}^{3} & 5x_{n+8}^{4} & 6x_{n+8}^{5} & 7x_{n+8}^{6} & 8x_{n+8}^{7} & 9x_{n+8}^{8} & 10x_{n+8}^{9} & 11x_{n+8}^{10} \\
0 & 1 & 2x_{n+\frac{17}{2}} & 3x_{n+\frac{17}{2}}^{2} & 4x_{n+\frac{17}{2}}^{3} & 5x_{n+\frac{17}{2}}^{4} & 6x_{n+\frac{17}{2}}^{5} & 7x_{n+\frac{17}{2}}^{6} & 8x_{n+\frac{17}{2}}^{7} & 9x_{n+\frac{17}{2}}^{8} & 10x_{n+\frac{17}{2}}^{9} & 11x_{n+\frac{17}{2}}^{10} \\
0 & 1 & 2x_{n+9} & 3x_{n+9}^{2} & 4x_{n+9}^{3} & 5x_{n+9}^{4} & 6x_{n+9}^{5} & 7x_{n+9}^{6} & 8x_{n+9}^{7} & 9x_{n+9}^{8} & 10x_{n+9}^{9} & 11x_{n+9}^{10}
\end{bmatrix} \tag{11}$$

The continuous coefficients of the method (10) are similarly obtain from the inverse of (11) as:

$$\begin{aligned}
&\alpha_3(x) = 1 \\
&\beta_0(x) = \left(-\frac{4581629\ \ h}{16755200} + \mu - \frac{126233\ \ \mu^2}{85680\ \ h} + \frac{610807\ \ \mu^3}{514080\ \ h^2} - \frac{366199\ \ \mu^4}{616896\ \ h^3} + \frac{1205177\ \ \mu^5}{6168960\ \ h^4} - \frac{76871\ \ \mu^6}{1762560\ \ h^5} + \right. \\
&\qquad \left. \frac{3419\ \ \mu^7}{514080\ \ h^6} - \frac{1123\ \ \mu^8}{1645056\ \ h^7} + \frac{167\ \ \mu^9}{3701376\ \ h^8} - \frac{107\ \ \mu^{10}}{61689600\ \ h^9} + \frac{\mu^{11}}{33929280\ \ h^{10}}\right) \\
&\beta_1(x) = \left(-\frac{1693113\ \ h}{985600} + \frac{51\,\mu^2}{10\,h} - \frac{83393\ \ \mu^3}{12600\ \ h^2} + \frac{7921\ \ \mu^4}{1920\ \ h^3} - \frac{9335\ \ \mu^5}{6048\ \ h^4} + \frac{12937\ \ \mu^6}{34560\ \ h^5} \right. \\
&\qquad \left. - \frac{677\ \ \mu^7}{11200\ \ h^6} + \frac{149\ \ \mu^8}{23040\ \ h^7} - \frac{\mu^9}{2268\ \ h^8} + \frac{\mu^{10}}{57600\ \ h^9} - \frac{\mu^{11}}{3326400}\right) \\
&\beta_2(x) = \left(\frac{407367\ \ h}{8008000} - \frac{153\ \ \mu^2}{13\,h} + \frac{104813\ \ \mu^3}{5460\ \ h^2} - \frac{401181\ \ h^4}{29120\ \ h^3} + \frac{3713351\ \ \mu^5}{655200\ \ h^4} - \frac{165181\ \ \mu^6}{112320\ \ h^5} + \right. \\
&\qquad \left. \frac{8179\ \ \mu^7}{32760\ \ h^6} - \frac{1039\ \ \mu^8}{37440\ \ h^7} + \frac{2299\ \ \mu^9}{1179360\ \ h^8} - \frac{103\ \ \mu^{10}}{1310400\ \ h^9} + \frac{\mu^{11}}{720720\ \ h^{10}}\right) \\
&\beta_3(x) = \left(-\frac{4461461\ \ h}{1355200} + \frac{238\ \ \mu^2}{11\,h} - \frac{111953\ \ \mu^3}{2970\ \ h^2} + \frac{1384609\ \ h^4}{47520\ \ h^3} - \frac{759127\ \ \mu^5}{59400\ \ h^4} + \frac{332153\ \ \mu^6}{95040\ \ h^5} \right. \\
&\qquad \left. - \frac{8581\ \ \mu^7}{13860\ \ h^6} + \frac{2257\ \ \mu^8}{31680\ \ h^7} - \frac{367\ \ \mu^9}{71280\ \ h^8} + \frac{101\ \ \mu^{10}}{475200\ \ h^9} - \frac{\mu^{11}}{261360\ \ h^{10}}\right) \\
&\beta_4(x) = \left(\frac{376763\ \ h}{98560} - \frac{119\ \ \mu^2}{4\,h} + \frac{115523\ \ \mu^3}{2160\ \ h^2} - \frac{123821\ \ h^4}{2880\ \ h^3} + \frac{2547601\ \ \mu^5}{129600\ \ h^4} - \frac{96619\ \ \mu^6}{17280\ \ h^5} + \right. \\
&\qquad \left. \frac{3103\ \ \mu^7}{3024\ \ h^6} - \frac{1403\ \ \mu^8}{11520\ \ h^7} + \frac{703\ \ \mu^9}{77760\ \ h^8} - \frac{11\,\mu^{10}}{28800\ \ h^9} + \frac{\mu^{11}}{142560\ \ h^{10}}\right) \\
&\beta_5(x) = \left(-\frac{1882809\ \ h}{492800} + \frac{153\ \ \mu^2}{5\,h} - \frac{23533\ \ \mu^3}{420\ \ h^2} + \frac{20667\ \ h^4}{448\ \ h^3} - \frac{109279\ \ \mu^5}{5040\ \ h^4} + \frac{109723\ \ \mu^6}{17280\ \ h^5} - \right. \\
&\qquad \left. \frac{1727\ \ \mu^7}{1440\ \ h^6} + \frac{337\ \ \mu^8}{2304\ \ h^7} - \frac{101\ \ \mu^9}{9072\ \ h^8} + \frac{97\ \ \mu^{10}}{201600\ \ h^9} - \frac{\mu^{11}}{110880\ \ h^{10}}\right) \\
&\beta_6(x) = \left(\frac{360737\ \ h}{123200} - \frac{119\ \ \mu^2}{5\,h} + \frac{119093\ \ \mu^3}{2700\ \ h^2} - \frac{318847\ \ \mu^4}{8640\ \ h^3} + \frac{381983\ \ \mu^5}{21600\ \ h^4} - \frac{45733\ \ \mu^6}{8640\ \ h^5} + \right. \\
&\qquad \left. \frac{12893\ \ \mu^7}{12600\ \ h^6} - \frac{23\,\mu^8}{180\ \ h^7} + \frac{43\,\mu^9}{4320\ \ h^8} - \frac{19\,\mu^{10}}{43200\ \ h^9} + \frac{\mu^{11}}{118800\ \ h^{10}}\right) \\
&\beta_7(x) = \left(-\frac{15627\ \ h}{8800} + \frac{102\,\mu^2}{7\,h} - \frac{17159\ \ \mu^3}{630\,h^2} + \frac{77453\ \ \mu^4}{3360\,h^3} - \frac{423559\ \ \mu^5}{37800\ \ h^4} + \frac{29467\ \ \mu^6}{8640\,h^5} - \right. \\
&\qquad \left. \frac{1693\,\mu^7}{2520\,h^6} + \frac{863\,\mu^8}{10080\ \ h^7} - \frac{103\,\mu^9}{15120\ \ h^8} + \frac{31\,\mu^{10}}{100800\ \ h^9} - \frac{\mu^{11}}{166320\ \ h^{10}}\right) \\
&\beta_8(x) = \left(\frac{1140903\ \ h}{985600} - \frac{153\,\mu^2}{16\,h} + \frac{60439\ \ \mu^3}{3360\,h^2} - \frac{2457\ \ \mu^4}{160\,h^3} + \frac{1521413\ \ \mu^5}{201600\ \ h^4} - \frac{80437\ \ \mu^6}{3456\ oh^5} + \right. \\
&\qquad \left. \frac{4693\ \ \mu^7}{10080\ \ h^6} - - \frac{1391\ \ \mu^8}{23040\ \ h^7} + \frac{1777\ \ \mu^9}{362880\ \ h^8} - \frac{13\,\mu^{10}}{57600\ \ h^9} + \frac{\mu^{11}}{221760\ \ h^{10}}\right)
\end{aligned} \tag{12}$$

$$\beta_{\frac{17}{2}}(x) = \left(-\frac{3046144\ h}{4679675} + \frac{65536\ \mu^2}{12155} - \frac{116801536\ \mu^3}{11486475\ h^2} + \frac{1334272\ \mu^4}{153153\ h^3} - \frac{148209664\ \mu^5}{34459425\ h^4} + \frac{9728\ \mu^6}{7293\ h^5} - \frac{3085312\ \mu^7}{11486475\ h^6} + \frac{256\ \mu^8}{7293\ h^7} - \frac{59392\ \mu^9}{20675655\ h^8} + \frac{512\ \mu^{10}}{3828825\ h^9} - \frac{1024\ \mu^{11}}{379053675\ h^{10}}\right)$$

$$\beta_9(x) = \left(\frac{22423\ h}{197120} - \frac{17\ \mu^2}{18\ h} + \frac{4499\ \mu^3}{2520\ h^2} - \frac{556819\ \mu^4}{362880\ h^3} + \frac{345019\ \mu^5}{453600\ h^4} - \frac{24581\ \mu^6}{103680\ h^5} + \frac{83\ \mu^7}{1728\ h^6} - \frac{437\ \mu^8}{69120\ h^7} + \frac{71\ \mu^9}{136080\ h^8} - \frac{89\ \mu^{10}}{3628800\ h^9} + \frac{\mu^{11}}{1995840\ h^{10}}\right) f_{n+9}$$

Substituting the continuous coefficients (12) into (10) produces the continuous form of the new MOBAM method for k = 9, $\mu = \frac{17}{2}$ as:

$$\begin{aligned}
y(\mu + x_n) = y_{n+3} + &\left(-\frac{4581629\ h}{16755200} + \mu - \frac{126233\ \mu^2}{85680\ h} + \frac{610807\ \mu^3}{514080\ h^2} - \frac{366199\ \mu^4}{616896\ h^3} + \frac{1205177\ \mu^5}{6168960\ h^4} - \frac{76871\ \mu^6}{1762560\ h^5} + \frac{3419\ \mu^7}{514080\ h^6} - \frac{1123\ \mu^8}{1645056\ h^7} + \frac{167\ \mu^9}{3701376\ h^8} - \frac{107\ \mu^{10}}{61689600\ h^9} + \frac{\mu^{11}}{33929280\ h^{10}}\right) f_n + \\
&\left(-\frac{1693113\ h}{985600} + \frac{51\ \mu^2}{10\ h} - \frac{83393\ \mu^3}{12600\ h^2} + \frac{7921\ \mu^4}{1920\ h^3} - \frac{9335\ \mu^5}{6048\ h^4} + \frac{12937\ \mu^6}{34560\ h^5} - \frac{677\ \mu^7}{11200\ h^6} + \frac{149\ \mu^8}{23040\ h^7} - \frac{\mu^9}{2268\ h^8} + \frac{\mu^{10}}{57600\ h^9} - \frac{\mu^{11}}{3326400}\right) f_{n+1} + \\
&\left(\frac{407367\ h}{8008000} - \frac{153\ \mu^2}{13\ h} + \frac{104813\ \mu^3}{5460\ h^2} - \frac{401181\ h^4}{29120\ h^3} + \frac{3713351\ \mu^5}{655200\ h^4} - \frac{165181\ \mu^6}{112320\ h^5} + \frac{8179\ \mu^7}{32760\ h^6} - \frac{1039\ \mu^8}{37440\ h^7} + \frac{2299\ \mu^9}{1179360\ h^8} - \frac{103\ \mu^{10}}{1310400\ h^9} + \frac{\mu^{11}}{720720\ h^{10}}\right) f_{n+2} + \\
&\left(-\frac{4461461\ h}{1355200} + \frac{238\ \mu^2}{11\ h} - \frac{111953\ \mu^3}{2970\ h^2} + \frac{1384609\ h^4}{47520\ h^3} - \frac{759127\ \mu^5}{59400\ h^4} + \frac{332153\ \mu^6}{95040\ h^5} - \frac{8581\ \mu^7}{13860\ h^6} + \frac{2257\ \mu^8}{31680\ h^7} - \frac{367\ \mu^9}{71280\ h^8} + \frac{101\ \mu^{10}}{475200\ h^9} - \frac{\mu^{11}}{261360\ h^{10}}\right) f_{n+3} + \\
&\left(\frac{376763\ h}{98560} - \frac{119\ \mu^2}{4\ h} + \frac{115523\ \mu^3}{2160\ h^2} - \frac{123821\ h^4}{2880\ h^3} + \frac{2547601\ \mu^5}{129600\ h^4} - \frac{96619\ \mu^6}{17280\ h^5} + \frac{3103\ \mu^7}{3024\ h^6} - \frac{1403\ \mu^8}{11520\ h^7} + \frac{703\ \mu^9}{77760\ h^8} - \frac{11\ \mu^{10}}{28800\ h^9} + \frac{\mu^{11}}{142560\ h^{10}}\right) f_{n+4} + \\
&\left(-\frac{1882809\ h}{492800} + \frac{153\ \mu^2}{5\ h} - \frac{23533\ \mu^3}{420\ h^2} + \frac{20667\ h^4}{448\ h^3} - \frac{109279\ \mu^5}{5040\ h^4} + \frac{109723\ \mu^6}{17280\ h^5} - \frac{1727\ \mu^7}{1440\ h^6} + \frac{337\ \mu^8}{2304\ h^7} - \frac{101\ \mu^9}{9072\ h^8} + \frac{97\ \mu^{10}}{201600\ h^9} - \frac{\mu^{11}}{110880\ h^{10}}\right) f_{n+5} + \\
&\left(\frac{360737\ h}{123200} - \frac{119\ \mu^2}{5\ h} + \frac{119093\ \mu^3}{2700\ h^2} - \frac{318847\ \mu^4}{8640\ h^3} + \frac{381983\ \mu^5}{21600\ h^4} - \frac{45733\ \mu^6}{8640\ h^5} + \frac{12893\ \mu^7}{12600\ h^6} - \frac{23\ \mu^8}{180\ h^7} + \frac{43\ \mu^9}{4320\ h^8} - \frac{19\ \mu^{10}}{43200\ h^9} + \frac{\mu^{11}}{118800\ h^{10}}\right) f_{n+6} + \\
&\left(-\frac{15627\ h}{8800} + \frac{102\ \mu^2}{7h} - \frac{17159\ \mu^3}{630\ h^2} + \frac{77453\ \mu^4}{3360\ h^3} - \frac{423559\ \mu^5}{37800\ h^4} + \frac{29467\ \mu^6}{8640\ h^5} - \frac{1693\ \mu^7}{2520\ h^6} + \frac{863\ \mu^8}{10080\ h^7} - \frac{103\ \mu^9}{15120\ h^8} + \frac{31\ \mu^{10}}{100800\ h^9} - \frac{\mu^{11}}{166320\ h^{10}}\right) f_{n+7} + \\
&+\left(\frac{1140903\ h}{985600} - \frac{153\ \mu^2}{16\ h} + \frac{60439\ \mu^3}{3360\ h^2} - \frac{2457\ \mu^4}{160\ h^3} + \frac{1521413\ \mu^5}{201600\ h^4} - \frac{80437\ \mu^6}{3456\ oh^5} + \frac{4693\ \mu^7}{10080\ h^6} - - \frac{1391\ \mu^8}{23040\ h^7} + \frac{1777\ \mu^9}{362880\ h^8} - \frac{13\ \mu^{10}}{57600\ h^9} + \frac{\mu^{11}}{221760\ h^{10}}\right) f_{n+8} + \\
&+\left(-\frac{3046144\ h}{4679675} + \frac{65536\ \mu^2}{12155} - \frac{116801536\ \mu^3}{11486475\ h^2} + \frac{1334272\ \mu^4}{153153\ h^3} - \frac{148209664\ \mu^5}{34459425\ h^4} + \frac{9728\ \mu^6}{7293\ h^5} - \frac{3085312\ \mu^7}{11486475\ h^6} + \frac{256\ \mu^8}{7293\ h^7} - \frac{59392\ \mu^9}{20675655\ h^8} + \frac{512\ \mu^{10}}{3828825\ h^9} - \frac{1024\ \mu^{11}}{379053675\ h^{10}}\right) f_{n+\frac{17}{2}} +
\end{aligned} \tag{13}$$

$$\left(\frac{22423\ h}{197120} - \frac{17\mu^2}{18h} + \frac{4499\ \mu^3}{2520\ h^2} - \frac{556819\ \mu^4}{362880\ h^3} + \frac{345019\ \mu^5}{453600\ h^4} - \frac{24581\ \mu^6}{103680\ h^5} + \frac{83\mu^7}{1728\ h^6} - \frac{437\ \mu^8}{69120\ h^7} + \frac{71\mu^9}{136080\ h^8} - \frac{89\mu^{10}}{3628800\ h^9} + \frac{\mu^{11}}{1995840\ h^{10}}\right) f_{n+9}$$

where, $\mu = x - x_n$ and $\mu \in [0, 9h]$. Evaluating the continuous scheme (13) at the following points $\mu = 0, h, 2h, 4h, 5h, 6h, 7h, 8h, \frac{17}{2}h, 9h$ yields the discrete members of the Modified Block Hybrid Adams Method (MOBHAM) used as block integrators as:

$$\begin{aligned}
y_n\text{-}y_{n+3} &= -\frac{h}{2395993600}(655172947\ f_n+4115957703\ f_{n+1}- \\
&1218842064\ f_{n+2}+7887863048\ f_{n+3}-9159108530\ f_{n+4}+9154217358\ f_{n+5}- \\
&7015613176\ f_{n+6}+4254794544\ f_{n+7}-2773535193\ f_{n+8}+1559625728\ f_{n+\frac{17}{2}}-272551565\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+1}\text{-}y_{n+3} &= \frac{h}{9097288200}(50461697\ f_n-3041910300\ f_{n+1}- \\
&1291966170\ 0f_{n+2}-462284706\ f_{n+3}-4173983242\ f_{n+4}+4401077538\ f_{n+5}- \\
&3373129188\ f_{n+6}+2030152410\ f_{n+7}-1312470159\ f_{n+8}+735182848\ f_{n+\frac{17}{2}}-128011598\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+2}\text{-}y_{n+3} &= -\frac{h}{5822264448\ 00}(607643465\ f_n-1062274313\ 1f_{n+1}+ \\
&2398729095\ 84f_{n+2}+4807837797\ 36f_{n+3}-2320345790\ 62f_{n+4}+1788398940\ 90f_{n+5}- \\
&1194381611\ 28f_{n+6}+6645579969\ 6f_{n+7}-4082743193\ 1f_{n+8}+2243126886\ 4f_{n+\frac{17}{2}}-3841935383\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+4}\text{-}y_{n+3} &= -\frac{h}{5822264448\ 00}(188536777\ f_n-2711763483\ f_{n+1}+ \\
&2133337588\ 8f_{n+2}-2727431949\ 84f_{n+3}-4130574965\ 66f_{n+4}+1332914334\ 18f_{n+5}- \\
&7417908669\ 6f_{n+6}+3782896603\ 2f_{n+7}-2207333657\ 1f_{n+8}+1190356582\ 4f_{n+\frac{17}{2}}-2007444439\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+5}\text{-}y_{n+3} &= -\frac{h}{9097288200}(1469039\ f_n-22958364\ f_{n+1}+ \\
&207240132\ f_{n+2}-3671758974\ f_{n+3}-1124291311\ 0f_{n+4}-3605498754\ f_{n+5}+ \\
&117533988\ f_{n+6}+49927878\ f_{n+7}-55696641\ f_{n+8}+34471936\ f_{n+\frac{17}{2}}-6393530\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+6}\text{-}y_{n+3} &= -\frac{h}{2395993600}(676819\ f_n-9663225\ f_{n+1}+ \\
&76071600\ f_{n+2}-1050296312\ f_{n+3}-2703306034\ f_{n+4}-2358687474\ f_{n+5}- \\
&1313459576\ f_{n+6}+235126320\ f_{n+7}-113048793\ f_{n+8}+58064896\ f_{n+\frac{17}{2}}-9459021\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+7}\text{-}y_{n+3} &= -\frac{h}{1137161025}(177320\ f_n-2851563\ f_{n+1}+ \\
&26317632\ f_{n+2}-463764522\ f_{n+3}-1372289776\ f_{n+4}-926283930\ f_{n+5}- \\
&1363849344\ f_{n+6}-476918442\ f_{n+7}+45099912\ f_{n+8}-16252928\ f_{n+\frac{17}{2}}+1971541\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+8}\text{-}y_{n+3} &= -\frac{5h}{2328905779\ 2}(1480765\ f_n-20558967\ f_{n+1}+ \\
&156227280\ f_{n+2}-2062826376\ f_{n+3}-5235912110\ f_{n+4}-4488330990\ f_{n+5}- \\
&4496046984\ f_{n+6}-5126687280\ f_{n+7}-2447435991\ f_{n+8}+489291776\ f_{n+\frac{17}{2}}-58258915\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{17}{2}}\text{-}y_{n+3} &= -\frac{11h}{2463635865\ 600}(60848359\ f_n - 868413975\ f_{n+1} + \\
&6868940100\ f_{n+2} - 9707855773\ 2f_{n+3} - 2565428319\ 74f_{n+4} - 2077870363\ 14f_{n+5} - \\
&2270095902\ 36f_{n+6} - 2325873349\ 80f_{n+7} - 1890175016\ 73f_{n+8} - 2707318374\ 4f_{n+\frac{17}{2}} - 783270631\ f_{n+9})
\end{aligned}$$

$$\begin{aligned}
y_{n+9}\text{-}y_{n+3} &= -\frac{h}{37437400}(15301\ f_n-204204\ f_{n+1}+ \\
&1460844\ f_{n+2}-17238442\ f_{n+3}-40641458\ f_{n+4}-38375766\ f_{n+5}- \\
&33324148\ f_{n+6}-44035134\ f_{n+7}-22688523\ f_{n+8}-23461888\ f_{n+\frac{17}{2}}-6130982\ f_{n+9})
\end{aligned} \tag{14}$$

STABILITY ANALYSIS OF THE NEW BLOCK METHODS

The block method (9) is represented by a matrix finite difference equation in block form as:

$$\mathrm{A}^{(1)}\mathrm{Y}_{w+1} = \mathrm{A}^{(0)}\mathrm{Y}_w + h\mathrm{B}^{(1)}\mathrm{F}_w \tag{15}$$

where,

$$Y_{w+1} = (y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7}, y_{n+8}, y_{n+9})^T$$
$$Y_w = (y_{n-8}, y_{n-7}, y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n)^T$$
$$F_w = (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8}, f_{n+9})^T$$

for $w = 0, \ldots$ and $n = 0, 9, \ldots, N\text{-}9$ and the matrices $\mathrm{A}^{(1)}, \mathrm{A}^{(0)}, \mathrm{B}^{(1)}$ being 9 by 9 matrices whose entries are given by the coefficients of (9) and are as defined in equation (16) below:

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} \frac{147429}{89600} & \frac{-279}{1600} & \frac{13273}{5600} & \frac{-19107}{8960} & \frac{73809}{44800} & \frac{-5039}{5600} & \frac{3663}{11200} & \frac{-909}{12800} & \frac{25}{3584} \\ \frac{-19469}{56700} & \frac{-7817}{5670} & \frac{-9371}{56700} & \frac{-14117}{56700} & \frac{12133}{56700} & \frac{-3373}{28350} & \frac{491}{11340} & \frac{-1063}{113400} & \frac{13}{14175} \\ \frac{163531}{7257600} & \frac{-391711}{907200} & \frac{-349817}{453600} & \frac{1083167}{3628800} & \frac{-129581}{725760} & \frac{38599}{453600} & \frac{-25759}{907200} & \frac{42187}{7257600} & \frac{-7}{12800} \\ \frac{10063}{1451520} & \frac{-42767}{907200} & \frac{225623}{453600} & \frac{2381791}{3628800} & \frac{-583073}{3628800} & \frac{5779}{90720} & \frac{-17663}{907200} & \frac{27467}{7257600} & \frac{-2497}{7257600} \\ \frac{167}{56700} & \frac{-701}{28350} & \frac{23189}{56700} & \frac{13903}{11340} & \frac{23189}{56700} & \frac{-701}{28350} & \frac{167}{56700} & \frac{-23}{113400} & 0 \\ \frac{603}{89600} & \frac{-99}{2240} & \frac{2647}{5600} & \frac{47727}{44800} & \frac{47727}{44800} & \frac{2647}{5600} & \frac{-99}{2240} & \frac{603}{89600} & \frac{-7}{12800} \\ \frac{13}{14175} & \frac{-32}{2025} & \frac{5494}{14175} & \frac{17632}{14175} & \frac{2174}{2835} & \frac{17632}{14175} & \frac{5494}{14175} & \frac{-32}{2025} & \frac{13}{14175} \\ \frac{4675}{290304} & \frac{-3175}{36288} & \frac{10735}{18144} & \frac{123575}{145152} & \frac{190775}{145152} & \frac{11575}{18144} & \frac{48425}{36288} & \frac{101635}{290304} & \frac{-25}{3584} \\ \frac{-9}{140} & \frac{99}{350} & \frac{-299}{700} & \frac{1899}{700} & \frac{-747}{700} & \frac{199}{70} & \frac{-153}{700} & \frac{2313}{1400} & \frac{7}{25} \end{bmatrix} \quad (16)$$

where, $w = 0,1,2,\ldots 9$ and n is the grid index.

Zero-stability: Zero-stability is concerned with the stability of the difference system (15) as h tends to zero. Thus, as $h \to 0$, the method (15) tends to the difference system:

$$\mathrm{A}^{(1)}\mathrm{Y}_{w+1} = \mathrm{A}^{(0)}\mathrm{Y}_w \quad (17)$$

whose first characteristic polynomial $\rho(\lambda)$ is given by:

$$\rho(\lambda) = \left|\lambda \mathrm{A}^{(0)} - A^{(1)}\right| \quad (18)$$

Substituting $A^{(0)}$, $A^{(1)}$ from (16) into the characteristics polynomial $\rho(\lambda)$ (18) gives:

$$\rho(\lambda) = \left|\lambda \mathrm{A}^{(0)} - A^{(1)}\right| = \lambda^8(\lambda - 1) \quad (19)$$

By Fatunla (1991), the block method (9) is zero-stable since in (19), $\rho(\lambda) = 0$ satisfies $|\lambda_j| \le 1, j = 1, \ldots$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 1. The block method is therefore convergent according to

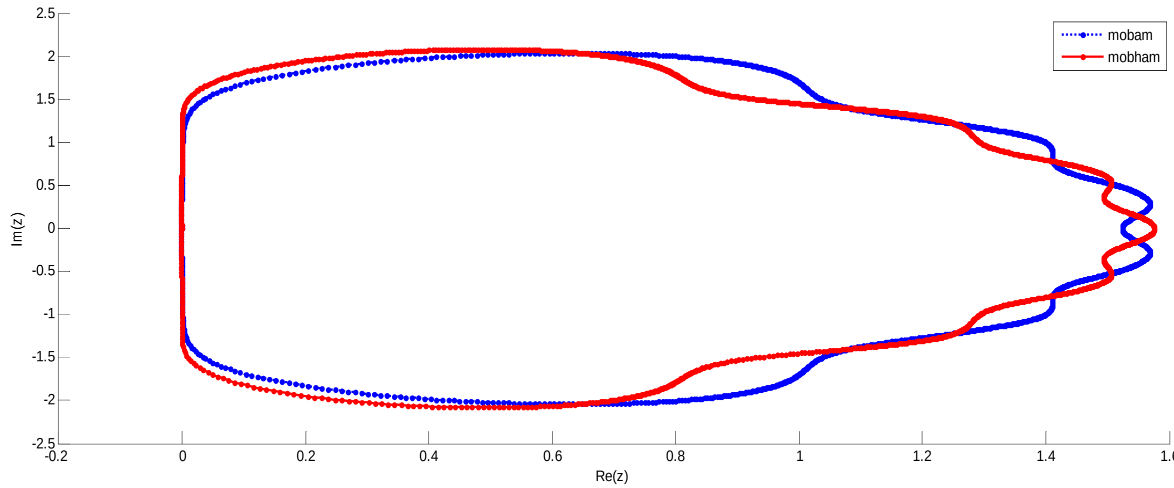

Fig. 1: Absolute stability regions of the MOBAM methods

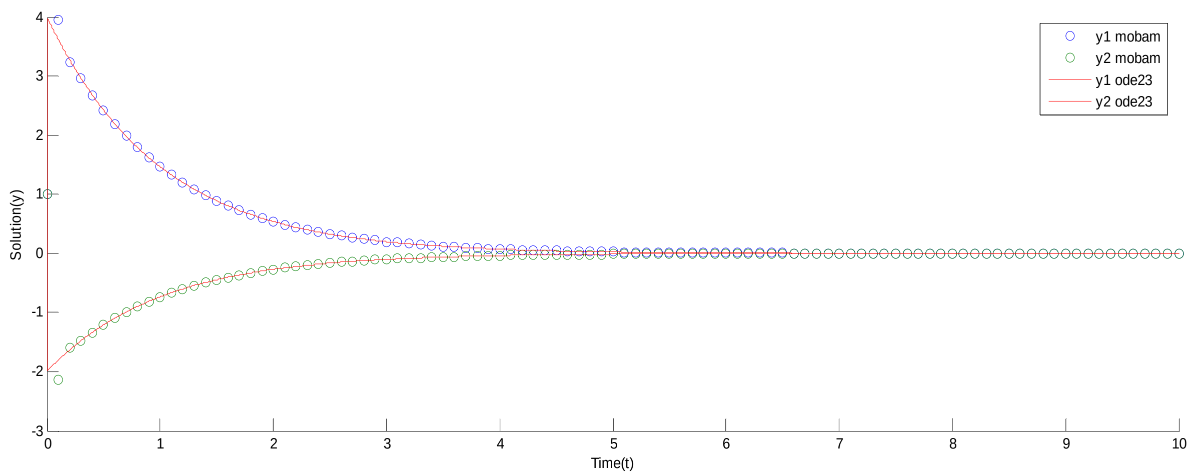

Fig. 2: Solution of example 1 using (10) and ode23

Henrici (1962). Since the MOBAM method is both consistent and zero-stable, it is convergent. Using the same procedure, the MOBHAM method in (14) is also zero-stable and consistent, hence convergent.

Order of the MOBAM methods: The orders and error constants of the new block methods are obtained using Chollom *et al*. (2007).

The block method (10) is of uniform order $(10,10,10,10,10,10,10,10,10)^T$ with error constants:

$$c_{11}=\left(-\frac{11899}{1971200},-\frac{5609}{7484400},\frac{171137}{479001600},\frac{90817}{479001600},\frac{263}{7484400},\frac{443}{1971200},-\frac{62}{467775},\frac{18665}{19160064},-\frac{179}{30800}\right)^T$$

while the block method (15) is of uniform order $(11,11,11,11,11,11,11,11,11,11)^T$ with error constants:

$$c_{12}=\left(\frac{24307}{5420800},\frac{28249}{54885600},-\frac{36919}{188179200},-\frac{37987}{439084800},-\frac{4997}{164656800},-\frac{443}{5420800},-\frac{31}{1715175},-\frac{5905}{52690176},-\frac{4908607}{5945425920},-\frac{17}{96800}\right)^T$$

Absolute stability region of the MOBAM methods: Following Chollom *et al*. (2007) and Butcher (1985), the block methods (9) and (14) are reformulated as General linear methods and the region of absolute stability of the method was plotted using the matlab program and is shown in Fig. 1.

Figure 1 reveals that the MOBAM methods are A-stable.

NUMERICAL EXPERIMENTS

In this section, the MOBAM methods (9) and (14) are tested on linear and nonlinear stiff initial value problems of ODEs. The solution curves obtained for the MOBAM methods are compared with the well-known MATLAB ODE 23 solver.

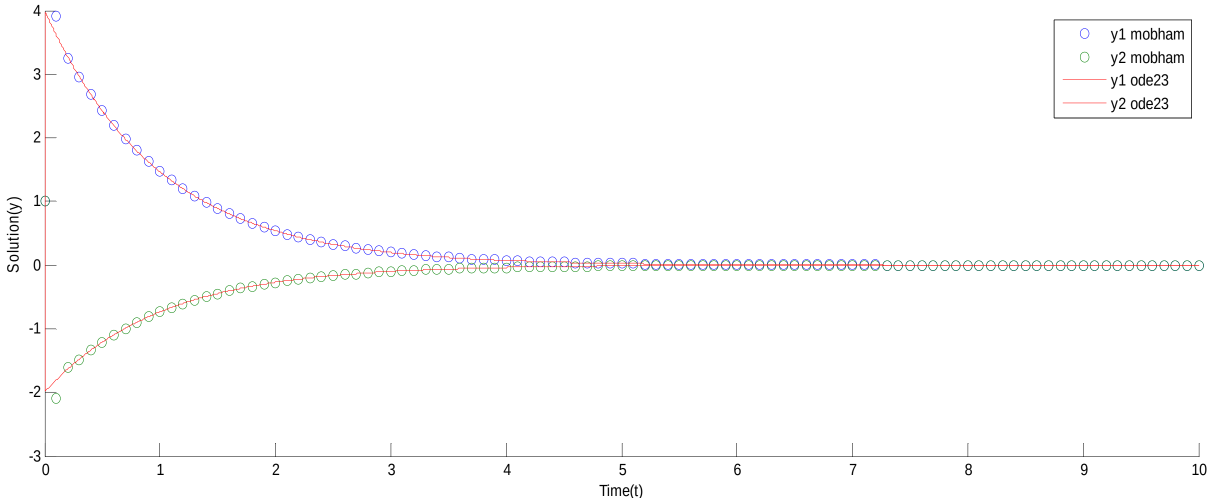

Fig. 3: Solution of example 1 using (15) and ode23

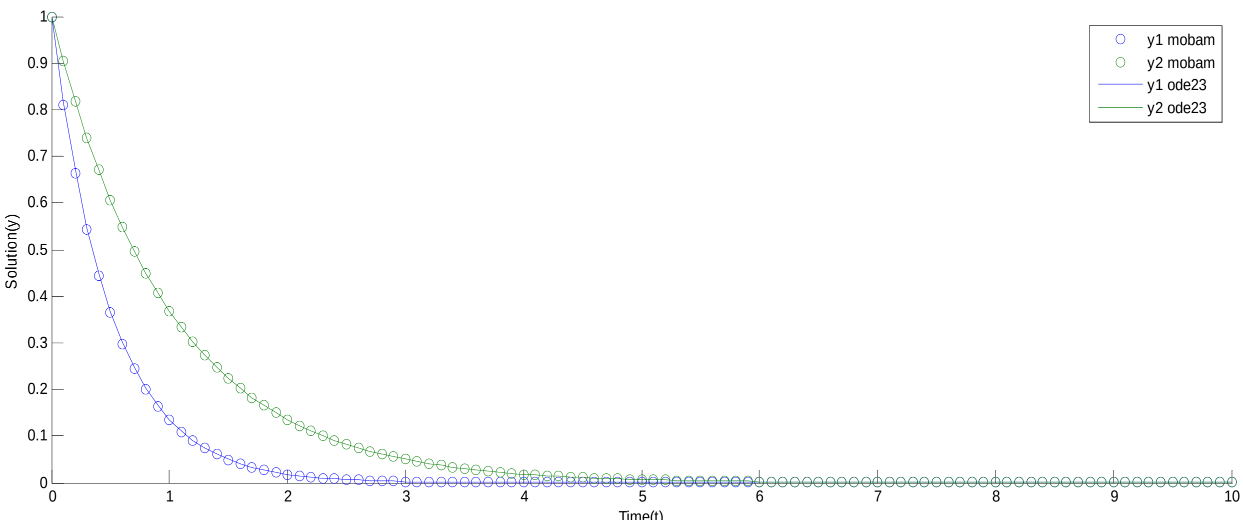

Fig. 4: Solution of example 2 using (10) and ode23

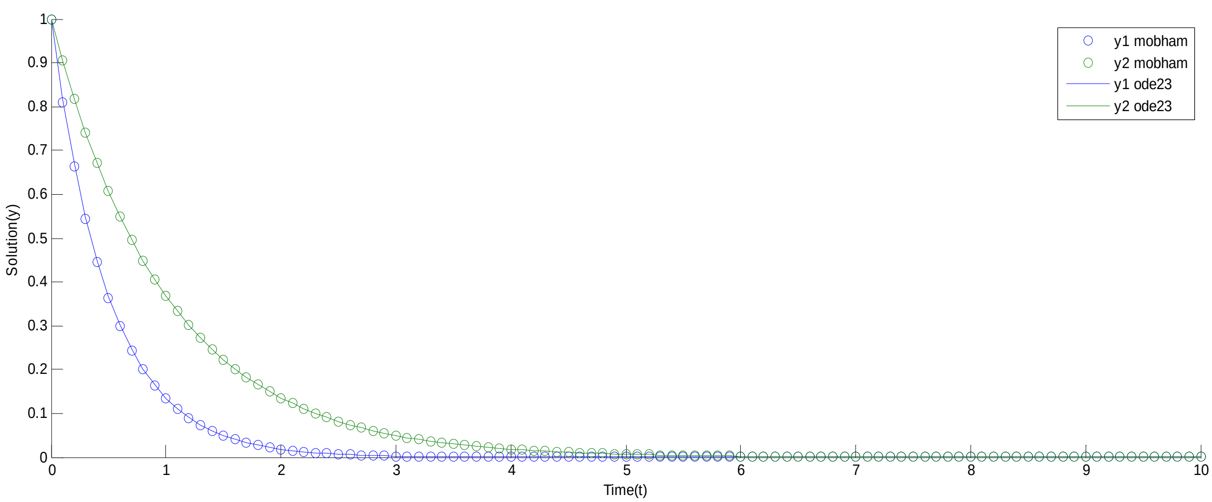

Fig. 5: Solution of example 2 using (15) and ode23

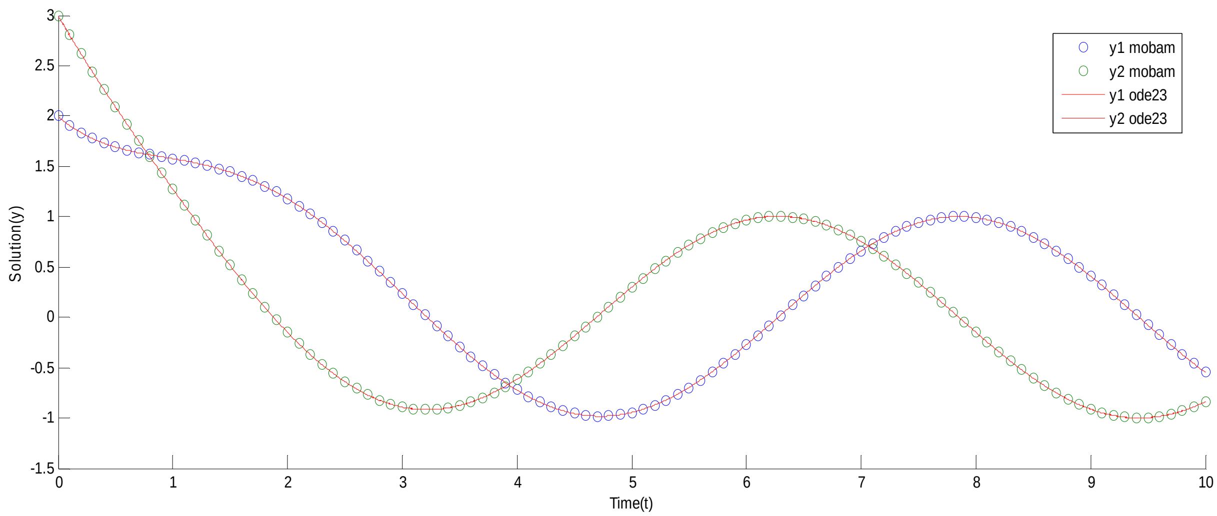

Fig. 6: Solution of example 3 using (10) and ode23

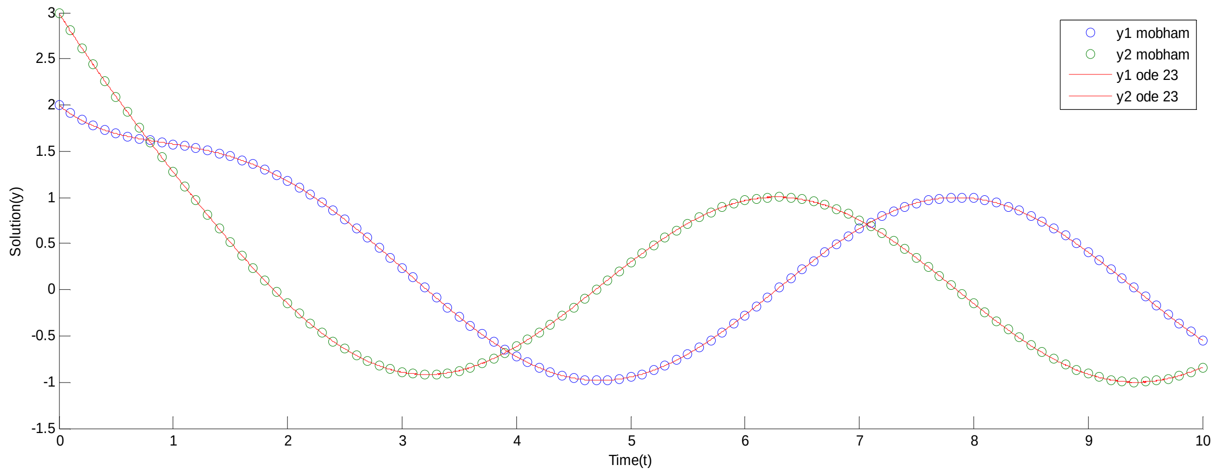

Fig. 7: Solution of example 3 using (15) and ode23

Example 1: We consider a well -known classical system experimented in Baker (1989), Stabrowski (1997), Vigo-Aguiar and Ramos (2007) and Akinfenwa *et al*. (2011), in the range $0 \le t \le 10$:

$$y_1' = 998\ y_1 + 1998\ y_2, \quad y_1(0) = 1$$
$$y_2' = -999\ y_1 - 1999\ y_2, \quad y_1(0) = 1$$

Its exact solution is given by the sum of two decaying exponential components:

$$y_1 = 4e^{-t} - 3e^{-1000t}, \quad y_1 = -2e^{-t} + 3e^{-1000t}$$

The stiffness ratio is 1:1000, h = 0.1 (Fig. 2 and 3)

Example 2: Consider the stiffly nonlinear problem which was proposed by Kaps *et al*. (1981) in the range $0 \le t \le 10$.

$$y_1' = -(\varepsilon^{-1} + 2)y_1 + \varepsilon^{-1} y_2^2, \quad y_1(0) = 1$$
$$y_2' = y_1 - y_2 - y_2^2, \quad y_1(0) = 1$$

The smaller is the ε, the more serious the stiffness of the system. Its exact solution is given by (Fig. 4 and 5):

$$y_1 = y_2^2,\ y_2 = e^{-t},\ \text{h} = 0.1,\ \varepsilon = 10^{-8}$$

Example 3: Consider the stiff problem; see Brugnano and Trigiante (1998):

$$y_1' = -2y_1 + y_2 + 2\sin x, \quad y_1(0) = 2$$
$$y_2' = 998y_1 - 999y_2 + 999(\cos x - \sin x), \quad y_1(0) = 3$$

The stiffness ratio of the problem is 1000. h = 0.1, $0 \le x \le 10$. Its exact solution is given by (Fig. 6 and 7):

$y_1(x) = 2e^{-x} + \sin x,\ y_2(x) = 2e^{-x} + \cos x$

CONCLUSION

New MOBAM methods have been proposed and implemented as self-starting methods for the solution of stiff ordinary differential equations. The MOBAM methods are A-stable making them suitable for the numerical solution of stiff problems. The accuracy and efficiency of the new block methods have been demonstrated on both linear and non-linear stiff problems as can be seen in Fig. 2 to 7. The authors in the next work will address the theoretical aspect and implementation strategies of the MOBAM class.

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