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Research Article

Estimate by the L^2 Norm of a Parameter Poisson Intensity Discontinuous

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Abstract: The model considered has two types of discontinuities. The first parameter is a parameter of scale; the second is a parameter of translation. The aim of study is to show that the minimum distance estimator is consistent and asymptotically normal. The problem of estimation studied in this work is the same time singular and regular. It is singular because the intensity is a discontinuous function and we obtains the consistency with the discontinuity and the study of asymptotic normality is based on the regularity of the function $\Lambda(\theta, t)$.

Keywords: Asymptotic properties, minimum distance estimator, parameter estimation, Poisson process, singular estimation

INTRODUCTION

In many areas of everyday life we use the inhomogeneous Poisson processes; there exists a large literature describing the different point processes and particularly Poisson with special attention to parameter estimation problems.

The properties of the estimators (Maximum Likelihood (MLE), Bayesian (BE), Minimum Distance (MDE) are usually studied in the case of smooth w.r.t., parameter situation i.e., the intensity function has one or two derivative, w.r.t. unknown parameter.

According to general theory all these estimators in regular (smooth) case are consistent and asymptotically normal see, Kutoyants (1998).

Moreover, the first two estimators are asymptotically efficient in usual sense.

In non regular case, when, say intensity function is discontinuous, the properties of estimators are quite different. Remember that the MDE is a particular case of the minimum contrast estimator if we consider the function $||X() - \Lambda()||$ as a contrast see Le (1972) for details. We have several possibilities of the choice of the space H.

In the case H = L^2 (μ) the measure μ can also be chosen in different ways continuous, discrete, etc.

Others definitions of the MDE can also be realized. Note that the asymptotic behavior of the estimator depends strongly of the chosen metric.

We are mainly interested in the properties of the MDE in the case H = L^2 (μ). Particularly we show that under regularity conditions the MDE are consistent, asymptotically normal. For another metric this estimator is also consistent but its limit distribution is not Gaussian (Kutoyants and Liese, 1992).

We did a study on Regular properties of the MDE for Non Smooth Model of Poisson Process (Ba and Dabye, 2009).

We also studied the asymptotic behaviour of the maximum likelihood estimator and the Bayesian estimator (Ba *et al*., 2008).

The present study is devoted to the dimensional case i.e., we suppose that the intensity function $S(\nu, t)$ depends to 2 dimensional parameter $\nu = (\gamma_1, \gamma_2)$. Our objective is to study the Minimum Distance Estimator (MDE) built from the norm L2. We show that the MDE is consistent and asymptotically normal. We studied the proprieties of the estimators using other norms and others models. Moreover in the case not hilbertien, it was shown in the studies of Kutoyants and Liese (1992) that we cannot have the asymptotic normality.

STATISTICAL MODEL

We observe n trajectories $X_j = (X_j(t), t \in [0,T])$ $j = 1, \dots, n$ of the Poisson process with intensity function $S(\nu_o, t) = f(\theta_1 t + \theta_2) + \lambda$.

Here $\nu_o = (\theta_1, \theta_2)$ in the unknown parameter, which we have to estimate, the function f and the constant λ are known and positive:

$$E_{\theta_o} X_j(t) = \Lambda(\theta_o, t) = \int_0^t S(\theta_o, s) ds \text{ with}$$
$$S(\theta_o, t) = f(\theta_1 t + \theta_2) + \lambda$$

Hypothesis A:

- The function $f()$ and the constant λ known and positive.
- The function $f()$ is continuously differentiable on $[a_1, \tau_1^*) \cup (\tau_1^*, \tau_2^*) \cup (\tau_2^*, \beta_1 T + \beta_2)]$, we suppose that $\beta_2 < T\alpha_1 + \alpha_2$.
- The function $f()$ have two jumps at $\tau_1^*, \tau_2^* \in (\beta_2, T\alpha_1 + \alpha_2)$ with $\tau_1^* < \tau_2^*$ and $f(\tau_i^* +) - f(\tau_i^* -) = r_i \neq 0$ $i = 1,2$.

We define this below the minimum distance estimator of θ_o and we are intereste to their properties when $n \longrightarrow \infty$.

Let $L^2([0,T])$ the Hilbert space with the $||h()|| = \left(\int_0^T h^2(t)\right)^{1/2}$, then we define the minimum distance estimator as solution of the equation:

$$\left|\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \Lambda(\theta_n^*,\cdot)\right|\right| = inf \left|\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \Lambda(\theta,\cdot)\right|\right|$$

STUDY OF THE CONSISTENCY

The following theorem ensures the consistency of the MDE.

Theorem 1: If the Hypothesis A is satisfied, then the estimator of the minimum distance θ_n^* is consistent: for all $\delta > 0$:

$$P_{\theta_o^{(n)}}\{\theta_n^* - \theta_o| > \delta\} \leq 4\,\frac{\int_0^T \Lambda(\theta_o,t)dt}{n\,\psi(\delta,\theta_o)^2} \longrightarrow 0$$

with $\psi(\delta,\theta_o) = \inf_{|\theta-\theta_o|>\delta} ||\Lambda(\theta,\cdot) - \Lambda(\theta_o,\cdot)||$

For the demonstration of this theorem, we need the following lemme.

Lemme: If the Hypothesis A is satisfied, then for all $\delta > 0$, the function $\psi(\delta,\theta_o) > 0$.

Demonstration: To show $\psi(\delta,\theta_o) > 0$, proceed by contradiction i.e., we suppose that for a $\delta > 0$, we have $\psi(\delta,\theta_o) = 0$.
In this case, it exist $\theta_o \neq \theta'_o$ such:

$$inf\ ||\Lambda(\theta,\cdot) - \Lambda(\theta_o)|| = ||\Lambda(\theta'_o) - \Lambda(\theta_o)|| = 0$$

so for all $t \in [0,T]$:

$$\Lambda(\theta'_o,t) = \Lambda(\theta_o,t)$$

since $\Lambda(\theta,t)$ is continuous function in t.

We deduce that $\frac{\partial}{\partial t}\Lambda(\theta'_o,t) = \frac{\partial}{\partial t}\Lambda(\theta_o,t)$ for all $t \in [0,T]$ i.e.:

$$\begin{aligned} f(\theta_1 t + \theta_2) + \lambda &= f(\theta'_1 t + \theta'_2) + \lambda \\ f(\theta_1 t + \theta_2) &= f(\theta'_1 t + \theta'_2) \end{aligned}$$

with means that:

$$\theta_1 ts_1 + \theta_1 = r_1 \theta_1 ts_2 + \theta_2 = r_2$$

we have:

$$\begin{pmatrix} ts_1 & 1 \\ ts_2 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

This leads to $\theta_1 = \theta'_1$ et $\theta_2 = \theta'_2$ i.e., $\theta_o = \theta'_o$ which contradicts the fact that $\theta_o \neq \theta'_o$.
So the lemma is proved.

Demonstration of the theorem: We put $Y_n(t) = \frac{1}{n}\sum_{j=1}^{n} X_j(t)$ and $Z_n(t) = \sqrt{n}\,(Y_n(t) - \Lambda(\theta_o,t))$which allows to write for all $\delta > 0$:

$$P_{\theta_o}^{(n)}\{||\theta_n^* - \theta_o|| > \delta\} \leq P_{\theta_o}^{(n)} \begin{Bmatrix} \inf\limits_{|\theta-\theta_o|\leq\delta} ||Y_n() - \Lambda(\theta,\cdot)|| > \\ \inf\limits_{|\theta-\theta_o|\leq\delta} ||Y_n() - \Lambda(\theta,)|| \end{Bmatrix}$$
$$\leq P_{\theta_o}^{(n)}\{2||Z_n()|| > \sqrt{n}\psi(\delta,\theta_o)\}$$

where, we have used the properties of the norm:

$$||Y_n() - \Lambda(\theta_o)|| + ||\Lambda(\theta_o,\cdot) - \Lambda(\theta,)||$$
$$\geq ||Y_n() - \Lambda(\theta,\cdot)||$$
$$||Y_n() - \Lambda(\theta_o,)|| - ||\Lambda(\theta_o,\cdot)|| \leq ||Y_n() - \Lambda(\theta,\cdot)||$$

Using the lemma $\psi(\delta,\theta_o) > 0$. so, applying the inequality of Chebychev it follows:

$$P_\theta^{(n)}\{2||Z_n()||\sqrt{n}\,\psi(\delta,\theta_o) \leq \frac{4E_{\theta_o}||Z_n()||^2}{n\psi(\delta,\theta_o)^2} \longrightarrow 0$$

Because,

$$E_{\theta_o}||Z_n()||^2$$
$$\int_0^T E_{\theta_o}\,(\sqrt{n}(Y_n(t) - \Lambda(\theta_o,t)))^2 dt$$
$$= \int_0^T E_{\theta_o}\left(\frac{1}{\sqrt{n}}\sum_1^n (Y_j(t) - \Lambda(\theta_o,t)\right)^2 dt$$
$$= \int_0^T (\Lambda(\theta_o,t)dt$$

The latter term being bounded, which demonstrated the theorem.

Asymptotic normality: In this section we will to prove the Asymptotic Normality of MDE.

We define the functions $F(\theta,t)$ et $G(\theta,t)$ for all $t \in [0,T]$ as:

$$F(\theta,t) = -\frac{1}{\theta_1}\left[\int_0^T g\,(\theta_1 s + \theta_2)ds - t\,g(\theta_1 t + \theta 2)\right.$$
et $G(\theta,t) = g(\theta_1 t + \theta_2) - g(\theta_2)$

Let us notice that the functions $R(\theta,t)$ et $G(\theta,t)$ are respectively the partial derives formal of the function $\Lambda(\theta,t)$.
Let us put $A_{1,1}(\theta) = \int_0^T F(\theta,t)^2 dt$:

$$\Lambda_{2,2}(\theta) = \int_0^T G(\theta,t)^2 dt$$
$$A_{12} = A_{2,1} = \int_0^T F\,(\theta,t)\,G(\theta,t)dt$$

and

$$C_{11}(\theta) = \int_0^T \int_0^T F(\theta,t)F(\theta,s)\Lambda(\theta, s\wedge t)\, dtds$$
$$C_{2,2}(\theta) = \int_0^T \int_0^T G(\theta,t)G(\theta,s)\Lambda(\theta, s\wedge t)\, dtds$$
$$C_{1,2}(\theta) = C_{2,1}(\theta) = \int_0^T \int_0^T F(\theta,t)G(\theta,s)\Lambda(\theta, s\wedge t)\, dtds$$

Let be carrees matrix of order 2:

$$A(\theta) = \begin{pmatrix} A_{11}(\theta) & A_{1,2}(\theta) \\ A_{1,1}(\theta) & A_{2,1}(\theta) \end{pmatrix}$$

and

$$C(\theta) = \begin{pmatrix} C_{11}(\theta) & C_{12}(\theta) \\ C_{21}(\theta) & C_{2,2}(\theta) \end{pmatrix}$$

the determinant of the matrix $A(\theta)$ is:

$$det(A(\theta)) = \int_0^T F(\theta,t)^2 dt \int_0^T G(\theta,t)^2 dt - \left(\int_0^T F(\theta,t)G(\theta,t)dt\right)^2$$

To prove the asymptotic normality, we suppose that:

Hypothesis B:

$$A_{11}(\theta)\, A_{2,2}(\theta) \neq (A_{1,2}(\theta))^2$$

Théorème 2: If the Hypothesis A and B are satisfied then the minimum distance estimator is asymptotically normal i.e.:

$$\sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow N(0, \mathcal{A}^{-1}C(\theta_o)\mathcal{A}^{-1}(\theta_o))$$

Demonstration: We define the function $H_n(\theta)$ by:

$$H_n(\theta) = \int_0^T \left[\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta,t)\right]^2 dt$$

Let us remind that the minimum distance estimator θ_n^* is a solution of the equation:

$$\theta_n^* = arg \inf_{\theta = \Theta} H_n(\theta)$$

Let us note that $\theta_n^* = (\theta_{1,n}^*, \theta_{2,n}^*)$ and $\theta = (\theta_1, \theta_2)$ the function $H_n(\theta)$ is derivable.

So the MDE θ_n^* minimize the function $H_n(\theta)$ and is consistent in θ_o element of Θ.
It with a probability numeric.
This estimator is a solution of system:

$$\frac{\partial}{\partial\theta} H_n(\theta)_{\theta=\theta_1} = 0 \quad \frac{\partial}{\partial\theta} H_n(\theta)_{\theta=\theta_2} = 0$$

The calculation of the partial derives by report at θ_1 and θ_2 gives us:

$$\frac{\partial}{\partial\theta_1} H_n(\theta) = \frac{2}{\theta_1} \int_0^T \left[\int_0^t g(\theta_1 s + \theta_2)d\theta - tg(\theta_1 t + \theta_2)\right] \times \left[\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta,t)\right] dt$$

$$\frac{\partial}{\partial\theta_2} H_n(\theta) = -\frac{2}{\theta_1}\int_0^T [\, g(\theta_1 t + \theta_2 \; - g(\theta_2)]\, \left[\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(theta,t\, dt\right.$$

thus using (3) and (4), we obtain:

$$\int_0^T \left(\int_0^t g(\theta_{1,n}^* s + \theta_{2,n}^*)ds - tg(\theta_{1,n}^* t + \theta_{2,n}^*)\right) \left(\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta_n^*, t)\right) dt = 0$$

$$\int_0^T [g(\theta_{1,n}^* t + \theta_{2,n}^*) - g(\theta_{2,n}^*)] \left(\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta_n^*, t)\right) dt = 0$$

We can write this system as:

$$\int_0^T F(\theta_n^*, t)\left(\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta_o, t) + \Lambda(\theta_o, t) - \Lambda(\theta n*, t) dt = 0\right.$$

$$\int_0^T G(\theta_n^*, t)\left(\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta_o, t) + \Lambda(\theta_o, t) - \Lambda(\theta n*, t) dt = 0\right.$$

We introduce the process us $W_n()$ define by:

$$W_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^n (X_j(t) - \Lambda(\theta_o, t))$$

thus,

$$\int_0^T W_n(t)F(\theta_n^*, t)\, dt = \sqrt{n}\int_0^T F(\theta_n^*, t)[\Lambda(\theta_n^*, t) - \Lambda(\theta_o, t)\, dt$$

$$\int_0^T W_n(t)G(\theta_n^*, t)\, dt = \sqrt{n}\int_0^T G(\theta_n^*, t)[\Lambda(\theta_n^*, t) - \Lambda(\theta_o, t)\, dt$$

Then,

$$\int_0^T [\, \Lambda(\theta + h, t) - \Lambda(\theta, t) - (h, \Lambda(\theta, t) - (h, \dot{\Lambda}(\theta, t)^2 dt = \theta(|h|^2)$$

Using the continuity of this functions and the consistency of the estimator, we can write the system as:

$$\int_0^T W_n(t)F(\theta_o, t)dt = \int_0^T F(\theta_o, t)(u_n^*, \dot{\Lambda}(\theta_o, t)dt + \theta(1)$$

$$\int_0^T W_n(t)G(\theta_o,t)dt$$
$$= \int_0^T G(\theta_o,t)(u_n^*, \dot{\Lambda}(\theta_o,t))\, dt + \theta(1)$$

where we put $u_n^* = (u_{1n}^*, u^*2n)^t$, $u_{1,n}^* = \sqrt{n}(\theta^*1, n - \theta_1)$ et $u_{2,n}^* = (\sqrt{n}(\theta_{2,n}^* - \theta_2)$ we introduce the vector $Y_n = (Y_n^{(1)}, Y_n^{(2)})^T$
with,

$$Y_n^{(1)} = \int_0^T F(\theta_o,t)\, W_n(t)\, dt$$
$Y_n^{(2)} = \int_0^T G(\theta_o,t)\, W_n(t)\, dt$ we obtain
$$Y_n^{(1)} = A_{1,1}(\theta_o)u_{1,n}^*) + A_{1,2}(\theta_o)u_{2,n}^* + \theta(1)$$
$$Y_n^{(2)} = A_{2,1}(\theta_o)u_{1,n}^* t) + A_{2,2}(\theta_o) + \theta(1)$$

or under matrix shape:

$$A_n(\theta_o)\, u_n^* = Y_n + \theta(1)$$

From the hypothesis B, the matrix of order 2 $A(\theta_o)$ is invertible thus we can write $B(\theta_o) = A(\theta_o)^{-1}$, so:

$$u_n^* = B(\theta_o)Y_n + \theta(1)$$

We show that $n\, E_{\theta_o}(\theta_n^* - \theta_o)(\theta_n^* - \theta_o)^t = B(\theta_o)C(\theta_o)\, B(\theta_o)^t + \theta(1)$
We have:

$$\int_0^T F(\theta_o,t)W_n(t)dt$$
$$= \int_0^T \left(\frac{1}{\sqrt{n}}\sum_{j=1}^n (X_j(t) - \Lambda(\theta_o,t)\right) F(\theta_o,t)\, dt$$
$$\frac{1}{\sqrt{n}}\sum_{j=1}^n \int_0^T (X_j(t) - \Lambda(\theta_o,t)F(\theta_o,t)\, dt$$
$= \frac{1}{\sqrt{n}}\sum_{j=1}^n \psi_j^{(1)}$ with $\psi_j^{(1)}$
$$= \int_0^T (X_j(t) - \Lambda(\theta_o,t))F(\theta_o,t)dt$$

$$\int_0^T G(\theta_o,t)\, W_n(t)\, dt$$
$$= \int_0^T \left(\frac{1}{\sqrt{n}}\right)\sum_{j=1}^n (X_j(t) - \Lambda(\theta_o,t)\Big) F(\theta_o,t)\, dt$$
$$= \frac{1}{\sqrt{n}}\sum_{j=1}^n \int_0^T (X_j(t) - \Lambda(\theta_o,t)G(\theta_o,t)dt$$
$= \frac{1}{\sqrt{n}}\sum_{j=1}^n \psi_j^{(2)}$ with $\psi_j^{(1)}$
$$= \int_0^T (X_j(t) - \Lambda(\theta_o,t))G(\theta_o,t)dt$$
$$E_{\theta_o}\psi_j^{(1)}\psi_j^{(2)} = C_{\ell,i}(\theta_o)$$

Let us note that the terms of the matrix are:

$$Y^{(1)} = \int_0^T \left(\int_t^T F(\theta_o,s)\, ds\right)\, dW(\Lambda(\theta_o,t))$$

$$Y^{(2)} = \int_0^T \left(\int_t^T G(\theta_o,s)ds\right)\, dW\big(\Lambda(\theta_o,t)\big)$$

We have the following relation:

$$C_{\ell,i}(\theta_o) = E_{\theta_o}(Y^{(\ell)}Y^{(i)})$$

We obtain:

$$C_{1,1}(\theta_o) = \int_0^T \left(\int_t^T F(\theta_o,s)\, ds\right)^2 S(\theta_o,t)\, dt$$
$$C_{2,2}(\theta_o) = \int_0^T \left(\int_t^T G(\theta_o,s)\, ds\right)^2 S(\theta_o,t)\, dt$$

and,

$$C_{1,2}(\theta_o) = C_{2,1}(\theta_o) =$$
$$\int_0^T \left[\int_t^T F(\theta_o,s)\, ds \int_0^T G(\theta_o,r)\, dr\right] S(\theta_o,t)\, dt$$

$$\psi_1^{(1)} = \int_0^T [X_1(t) - \Lambda(\theta_o,t)]\, F(\theta_o,t)\, dt$$
$$= \int_0^T \int_s^T 1_{0\le s\le t}[dX_1(s) - S(\theta_o,t)\, ds]\, F(\theta_o,t)\, dt$$
$$= \int_0^T \int_0^T F(\theta_o,t)\, dt[dX_1(s) - S(\theta_o,t)\, ds]$$
$$= \int_0^T \int_s^T F(\theta_o,t)\, dtd\pi(s)$$

For $\psi_1^{(2)}$ we have:

$$\psi_1^{(2)} = \int_0^T \int_s^t G(\theta_o,t)dt\, d\pi(s)$$

On the other hand using:

$$E_{\theta_o}[X_1(t) - \Lambda(\theta_o,t)][X_1(s) - \Lambda(\theta_o,s) = \Lambda(\theta_o, t \wedge s]$$

we obtain for example:

$$E_{\theta_o}\psi_1^{(1)}\psi_1^{(2)} = \int_0^T \int_0^T F(\theta_o,t)G(\theta_o,s)\Lambda(\theta_o,t \wedge sds\, dt$$

Using the central theorem limit, we obtain:

$$Y_n \Longrightarrow \mathcal{N}(0, C(\theta_o))$$

as $\sqrt{n}(\theta_n^* - \theta_o) = B(\theta_o)\, Y_n + \theta(1)$

we deduce:

$u_n^* = \sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow \mathcal{N}(0, B(\theta_o)\, C(\theta_o)\, B(\theta_o)^t)$ i.e.:

$$\sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow \mathcal{N}(0, A^{-1}(\theta_o)\, C(\theta_o)\, A^{-1}(\theta_o)^t)$$

So the asymptotic normality is proved.

REFERENCES

Ba, D.B. and A.S. Dabye, 2009. On regular properties of the MDE for non smooth model of poisson process, Indian J. Math., 51(1): 145-162.

Ba, D.B., A.S. Dabye, A. Diakhaby and F.N. Diop, 2008. Comportment asymptotique des estimateurs de paramètre d'un processus de Poisson pour un modèle non régulier. Annales de l'Université Ndjamena, 3: 55-65.

Kutoyants, Y.A., 1998. Statistical Inference for Spatial Poisson Processes, Lect. Notes Statist. 134, Springer, New York, pp: 276.

Kutoyants, Y.A. and F. Liese, 1992. On minimum distance estimation for spatial poisson processes. Ann. Academial. Scient. Fennicae, ser. A. I., 17: 65-71.

Le, C., 1972. Limit of experiments. Proceeding of the 6th Berkeley Symposium on Mathematical Statistics and Probability,1: 245-261.