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Research Article

Study of an Estimate of the Minimum Distance for a Multidimensional Model of a Poisson Process

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Abstract: The aim of study is to show that the minimum distance estimator is consistent and asymptotically normal with the usual $\sqrt{n}$ rate of convergence for the intensty function of the process Poisson which have a particularty form. We consider the problem of estimation of a multi-dimensional parameter $\theta_o = (\omega_1^o, \dots, \omega_d^o, \gamma_1^o, \dots, \gamma_d^o)$. We suppose that the unknown parameter is 2d dimensional and the intensity function of the process is smooth the first d components and discontinuous the others d components of this parameter.

Keywords: Asymptotic normality, non regular model minimum distance estimation, parameter estimation, poisson processes

INTRODUCTION

We consider the problem of parameter estimation for a model 2d-dimensionnel of a Poisson process with discontinuous intensity function In homogeneous Poisson processes play on important role in applied problems. The wide choice of intensity functions allows obtaining a good fitting of the mathematical model to the real phenonemas. The behavior of the MDE in this discontinuous intensity function case is similar to that of the regular case, i.e., the rate of convergence is $\sqrt{n}$ and the estimator is asymptotically normal.

The properties of the estimators (Maximum Likelihood (MLE), Bayesian (BE), Minimum Distance (MDE) are usually studied in the case of smooth w.r.t., parameter situation i.e., the intensity function has one or two derivative, w.r.t., unknown parameter.

According to general theory all these estimators in regular (smooth) case are consistent and asymptotically normal (Kutoyants, 1998).

Moreover, the first two estimators are asymptotically efficient in usual sense.

In non regular case, when, say intensity function is discontinuous, the properties of estimators are quite different. Remember that the MDE is a particular case of the minimum contrast estimator if we consider the function $||X() - \Lambda()||$ as a contrast (Le, 1972) for details. We have several possibilities of the choice of the space H.

In the case H = L^2 (μ) the measure μ can also be chosen in different ways continuous, discrete, etc.

Others definitions of the MDE can also be realized. Note that the asymptotic behavior of the estimator depends strongly of the chosen metric.

We are mainly interested in the properties of the MDE in the case H = L^2 (μ). Particularly we show that under regularity conditions the MDE are consistent, asymptotically normal. For another metric this estimator is also consistent but its limit distribution is not Gaussian (Kutoyants and Liese, 1992).

We did a study on Regular properties of the MDE for Non Smooth Model of Poisson Process (Ba and Dabye, 2009).

We also studied the asymptotic behaviour of the maximum likelihood estimator and the Bayesian estimator (Ba and Dabye, 2008).

The present study is denoted to the multidimensional case for the model of intensity function:

$$S(\theta_o, t) = \sum_{i=1}^{d} g_i\,(\omega_i^o\; t + \gamma_i^o) + \lambda$$

Here, $\theta_o = (\omega_1^o, \dots, \omega_d^o, \gamma_1^o, \dots, \gamma_d^o)$ is the unknown parameter, which we have to estimate, the functions $g_i(\,)$ and the constant γ are known and positive.

METHODOLOGY

Preliminaries: We observe n trajectories $X_j = (X_j(t), t \in [0,T])$ $j = 1, \dots, n$ of the Poisson process with intensity function $S(\theta_o, t = \sum_{i=1}^{n} g_i(\omega_i^o t + \gamma_i^o) + \lambda$.

Here $\theta_o = (\theta_1^o, \dots, \theta_d^o, \gamma_1^o, \dots, \gamma_d^o)$ is the unknown parameter which we have to estimate. The functions $g_i(\,)$ and the constant λ are known and positive. The parameter $\theta_o \in \Theta = \otimes\, (\Gamma_i \times \Omega_i) \subset R^{2d}$ where $\Gamma_i = (\alpha_i, \beta_i)$ and $\Omega_i = (\alpha_{i+d}, \beta_{i+d})$.

Hypothesis $\boldsymbol{B_o}$:

- The function $g_i(\,)$ is continuously differentiable on semi-open intervals:

$[\alpha_{i+1}, \tau_i^*) \cup (\tau_i^*, \tau_{i+1}) \cup (\tau_{i+1}^*, \beta_i T + \beta_{i+1}]$

We suppose that $\beta_{i+1} < T\alpha_i + \alpha_{i+1}$.

- The function $g_i(\ )$ admit the finite jumps at the points τ_i^* with:

$$g_i(\tau_i^*+) - g(\tau_i^*-) = r_i > 0 \text{ and } g_i(\tau_i^*)g_i(\tau_i^*) > 0$$

By this condition all functions g_i are discontinuous on the interval of observations and these jumps take place on disjoint intervals.

We define this below the minimum distance estimator of θ_o and we are interested to their properties.

Let $L^2([0,T])$ the Hilbert space with the $\| h(\ ) \| = \left(\int_0^T h^2(t)\right)^{1/2}$.

We define the minimum distance estimator as solution of the equation:

$$\left\|\frac{1}{n}\sum_{j=1}^{n} X_j - \Lambda(\theta_n^*,.)\right\| = inf \left\|\frac{1}{n}\sum_{j=1}^{n} X_j - \Lambda(\theta_n,.)\right\|$$

Our goal is to study the consistency and the asymptotic properties of MDE (Minimum Distance Estimator) of θ_o.

Study of this consistency: We start with the following lemma:

Lemma 1: If the hypothesis B is satisfied, then for every $\delta > 0$ $\psi(\delta, \theta_o) = \inf_{|\theta-\theta_o|>\delta} \| \Lambda(\theta,.) - \Lambda(\theta_o,.) \| > 0$.

Demonstration: To show $\psi(\delta, \theta_o) > 0$, proceed by contradiction i.e., we suppose that for some $\delta > 0$, we have $\psi(\delta, \theta_o) = 0$.

In this case, it exist $\theta_0 \neq \theta_o$ such $\Lambda(\theta_1, t) = \Lambda(\theta_o, t)$. This leads to $\sum_{i=1}^{d} g_i(\omega_i^o t + \gamma_i^o) + \lambda = \sum_{i=1}^{d} g_i(\omega_i^1 t + \gamma_i^1) + \lambda$ foralmost all $t \in [0,T]$ except the points of discontinuity. The functions:

$$h_o(t) = \sum_{i=1}^{d} g_i(\omega_i^o t + \gamma_i^o) \ and \ h_1(t) = \sum_{i=1}^{d} g_i(\omega_i^1 t + \gamma_i^1)$$

$$h_o(t) = h_1(t) \Rightarrow \sum_{i=1}^{d} g_i(\omega_i^o t + \gamma_i^o) - g_i(\omega_i^1 t + \gamma_i^1) = 0$$

This equality and uniqueness of the discontinuity of each function $g_i(\ )$ leads to:

$$g_i(\omega_i^o t + \gamma_i^o) = g_i(\omega_i^1 t + \gamma_i^1)$$

Thus,

$$\omega_i^o ts_i + \theta_j = r_i \omega_i^1 ts_j + \theta_j = r_j$$

we have:

$$\begin{pmatrix} ts_i & 1 \\ ts_j & 1 \end{pmatrix} \begin{pmatrix} \theta_i \\ \theta_j \end{pmatrix} = \begin{pmatrix} r_i \\ r_j \end{pmatrix}$$

This leads to $\omega_1^o = \omega_1^1, \ldots, \omega_d^o = \omega_d^1$ et $\gamma_1^o = \gamma_1^1, \ldots, \gamma_d^o = \gamma_d^1$.

So the lemma is proved.

The following theorem ensures the consistency of the MDE.

Theorem 2: If the condition B is satisfied then the MDE θ_n^* is consistent for every $\delta > 0$ we have:

$$P_{\theta_o}^{(n)}\{|\theta_n^* - \theta_o| > \delta\} \leq \frac{4 \int_0^T \Lambda(\theta_o,t)}{n\,\psi(\delta,\theta_o)^2} \longrightarrow 0$$

Proof: Let us denote $Y_n(t) = \frac{1}{n}\sum_{j=1}^{n} X_j(t)$ and $Z_n(t) = \sqrt{n}(Y_n(t) - \Lambda(\theta_o, t))$.

This allows us to write for any $\delta > 0$:

$$P_{\theta_o}^{(n)}\{|\theta_n^* - \theta_o| > \delta\} \leq P_{\theta_n}^{(n)}\{\inf_{|\theta-\theta_o|\leq\delta} \| Y_n(.) - \Lambda(\theta,.) \| >$$

$$\inf_{|\theta-\theta_o|>\delta} \| Y_n(\ ) - \Lambda(\theta,) \| \leq P_{\theta_o}^{(n)}\{Z \| Z_n(\ ) \| > \sqrt{n}\psi(\delta, \theta_o)\}$$

where, we used the properties of the norm:

$$\| Y_n(\ ) - \Lambda(\theta_o,) \| + \| \Lambda(\theta_o,) - \Lambda(\theta,) \| \geq \| Y_n(\ ) - \Lambda(\theta,.) \|$$

and,

$$\| Y_n(\ ) - \Lambda(\theta_o,) \| - \| \Lambda(\theta_o,.) - \Lambda(\theta,.) \| \leq \| Y_n(\ ) - \Lambda(\theta,.) \|$$

and the fact that:

$$\inf_{|\theta-\theta_o|>\delta} \| \Lambda(\theta_o,) - \Lambda(\theta_o,.) \| = 0$$

According to the $\psi(\delta, \theta_o) > 0$. Therefore, by Chebychev inequality we have:

$$P_{\theta_o}^{(n)}\left\{2 \| Z_n(\ ) \| > \sqrt{n}\,\psi(\delta, \theta_o)\right\} \leq \frac{4\,E_{\theta_o}\|Z_n(\ )\|^2}{n\,\psi(\delta,\theta_o)} = \frac{4 \int_0^T \Lambda(\theta_o,t)\,dt}{n\,\psi(\delta,\theta_o)^2}$$

because $E_{\theta_o} Z_n^2(t) = \Lambda(\theta_o, t)$.

ASYMPTOTIC NORMALITY

To prove the asymptotic normality, we need the following conditions.

Hypotheses $\boldsymbol{B_1}$: The matrix $D(\theta)$ is nondegenerate uniformly in $\theta \in \Theta$ i.e., $\inf_{\theta\in\Theta} \inf_{|e|=1} e^T D(\theta) e > 0$.

Theorem 3: If the hypothesis B_o and B_1 are satisfied then the MDE θ_n^* is asymptotically normal:

$$\sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow \mathcal{N}\left(0, D^{-1}(\theta_o)R(\theta_o)D^{-1}(\theta_o)\right)$$

We need the following functions:

$$G_i(t,\theta) = \int_0^t [\, g_i(\omega_i s + \gamma_i) - g(\gamma_i)]\, ds$$

$$F_i(t,\theta) = \frac{\gamma_i}{\omega_i}\left(\int_0^t g_i(\omega_i s + \gamma_i) ds - t\, g_i(\omega_i t + \gamma_i)\right)$$

Note that the functions $G_i(\theta, t)$ and $F_i(\theta, t)$ are the formal partial derivatives of the function $\Lambda(\theta, t)$ with respect to γ_i and ω_i, respectively.
Introduce as well the square matrix $d \times d$:

$$A(\theta) = (A_{\ell,i}(\theta))_{1\le \ell,i\le d} B(\theta) = (B_{\ell,i}(\theta))_{1\le \ell,i\le d}$$

$$C(\theta) = (C_{\ell,i}(\theta))_{1\le \ell,i\le d}$$

Defined by their terms:

$$A_{\ell,i}(\theta) = \int_0^T G_\ell(t,\theta)\, G_i\,(t,\theta)\, dt$$

$$B_{\ell,i}(\theta) = \int_0^T F_\ell(t,\theta)\, G_i\,(t,\theta)\, dt$$

$$C_{\ell,i}(\theta) = \int_0^T F_\ell(t,\theta)\, F_i\,(t,\theta)\, dt$$

And the matrices $R^{p,q}(\theta_o) = (R_{\ell,i}^{p,q}(\theta_o))_{1\le \ell,i\le d} p, q = 1, 2$:

$$R_{\ell,i}^{(1,1)}(\theta_o) = \int_0^T \int_0^T G_\ell(t,\theta_o)\, G_i\,(t,\theta_o)\, \Lambda(\theta_o, t \wedge s)\, dt\, ds$$

$$R_{\ell,i}^{(1,2)}(\theta_o) = \int_0^T \int_0^T G_\ell(t,\theta_o)\, F_i\,(t,\theta_o)\, \Lambda(\theta_o, t \wedge s)\, dt\, ds$$

$$R_{\ell,i}^{(2,2)}(\theta_o) = \int_0^T \int_0^T F_\ell(t,\theta_o)\, F_i\,(t,\theta_o)\, \Lambda(\theta_o, t \wedge s)\, dt\, ds$$

Finally we introduce $2d \times 2d$ D and R:

$$D(\theta_o) = \begin{pmatrix} A(\theta_o) & B(\theta_o) \\ B(\theta_o) & C(\theta_o) \end{pmatrix}$$

$$R(\theta_o) = \begin{pmatrix} R^{(1,1)}(\theta_o) & R^{(1,2)}(\theta_o) \\ R^{(2,1)}(\theta_o) & R^{(2,2)}(\theta_o) \end{pmatrix}$$

We use the following lemma to prove the theorem.

Lemma 2: Under the hypothesis B_o the following relations hold:

$$\int_0^T G_\ell(t,\theta_n^*)\, G_i\,(t,\theta_n^*)\, dt = \int_0^T G_\ell(\theta_o,t)\, G_i(\theta_o,t)\, dt + 0(1)$$

$$\int_0^T F_\ell(t,\theta_n^*)\, F_i\,(t,\theta_n^*)\, dt = \int_0^T F_\ell(\theta_o,t) F_i(\theta_o,t) dt + O(1)$$

$$\int_0^T F_\ell(t,\theta_n^*)\, G_i\,(t,\theta_n^*)\, dt = \int_0^T F_\ell(\theta_o,t)\, G_i(\theta_o,t)\, dt + 0(1)$$

Lemma 3: If the hypothesis B_o and B_1 are satisfied then there exists a constant $k > 0$ such that for every $h \in \mathbb{R}^{2d}, \theta_o + h \in \Theta$, we have:

$$\parallel \Lambda(\theta_o + h) - \Lambda(\theta_o) \parallel \ge k|.h$$

Lemma 4: If the B_o and B_1 are satisfied, then for any $\nu \in (0,1/2)$ we have:

$$P_{\theta_o}^{(n)}\left\{|\theta_n^* - \theta_o| \le \frac{1}{n^\nu}\right\} \underset{n\to\infty}{\longrightarrow} 1$$

Introduce the process $W_n(\cdot)$ defined by:

$$W_1(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^n (X_j(t) - \Lambda(\theta_o, t))$$

Lemma 5:

$$\begin{pmatrix} \int_0^T W_n\,(t) G_1(t,\theta_n^*) dt \\ \vdots \\ \int_0^T W_n\,(t) G_d(t,\theta_n^*) dt \end{pmatrix} = \begin{pmatrix} \int_0^T W_n\,(t) G_1(t,\theta_o) dt \\ \vdots \\ \int_0^T W_n\,(t) G_d(t,\theta_o) dt \end{pmatrix} + 0(1)$$

$$\begin{pmatrix} \int_0^T W_n\,(t) F_1(t,\theta_n^*) dt \\ \vdots \\ \int_0^T W_n\,(t) F_d(t,\theta_n^*) dt \end{pmatrix} = \begin{pmatrix} \int_0^T W_n\,(t) F_1(t,\theta_o) dt \\ \vdots \\ \int_0^T W_n\,(t) F_d(t,\theta_o) dt \end{pmatrix} + 0(1)$$

Proof of theorem: Introduce the function $H_n(\theta)$ defined tel que:

$$H_n(\theta) = \int_0^T \left[\frac{1}{n}\sum_{j=1}^n X_j(t) - \Lambda(\theta, t)\right]^2 dt$$

and remind that the MDE $\theta_n^* = (\omega_{1,n}^*, \ldots, \omega_{d,n}^*, \gamma_{1,n}^*, \ldots, \gamma_{d,n}^*)$ is given by $\theta_n^* = arg\ \inf_{\theta\in\Theta} H_n(\theta)$.

As the function $H_n(\theta)$ is smooth with respect to θ and θ_n^* is extreme point, thus, the MDE θ_n^* is solution of the following system:

$$\left\{\frac{\partial}{\partial \omega_1} H_n(\theta)\right|_{\theta=\theta_n} = 0$$

$$\frac{\partial}{\partial \omega_d} H_n(\theta)_{\theta=\theta_n} = 0 \quad (1)$$

$$\left\{\frac{\partial}{\partial \gamma_1}\right|_{\theta=\theta_n} = 0$$

$$\frac{\partial}{\partial \gamma_2} H_n(\theta)_{\theta=\theta_n} = 0 \quad (2)$$

Note that as the MDE is consistent, the probability that is takes values on the border of the parameter espace is asymptotically negligible.
The calculation of the partial derivatives of:

$$\Lambda(\theta, t) = \sum_{i=1}^{d} \int_0^T g\,(\omega_i t + \gamma_i) dt + \lambda t \quad 0 \le t \le T$$

With respect to first d variables γ_i gives us:

$$\begin{cases} \frac{\partial}{\partial \gamma_1} \Lambda(\theta) = \frac{1}{\omega_1} \int_0^t [g_1(\omega_d s + \gamma_d) - g(\gamma_1)]\, ds = C_1(t,\theta) \\ \vdots \\ \frac{\partial}{\partial \gamma_d} \Lambda(\theta) = \frac{1}{\omega_d} \int_0^t [g_d(\omega_d s + \gamma_d) - g(\gamma_d)]\, ds = C_d(t,\theta) \end{cases}$$

The formal similar calculation for the other d variables ω_i leads to the relations:

$$\begin{cases} \frac{\partial}{\partial \omega_1} \Lambda(\theta) = \frac{\gamma_1}{\omega_1} \left(\int_0^t [g_1(\omega_1 s + \gamma_1) - g(\gamma_1)]\, ds - t g_1(\omega_1 t + \gamma_1) \right) \\ \vdots \\ \frac{\partial}{\partial \omega_d} \Lambda(\theta) = \frac{\gamma_d}{\gamma_d} \left(\int_0^t [g_d(\omega_d s + \gamma_d) ds - t g_d(\omega_d t + \gamma_d) \right) \end{cases}$$

$$\frac{\partial}{\partial \omega_1} \Lambda(\theta) = F_1(t,\theta) \quad \frac{\partial}{\partial \omega_d} \Lambda(\theta) = F_d(t,\theta)$$

Note that the functions $F_i(t,\theta)$ are discontinuous, but as dements of $L_2[0,T]$ they are continuous.
Introduce the vectors:

$u_n^* = (u_{1,n}^*, \ldots\ldots, u_{d,n}^*)$ and $v_n^* = (v_{1,n}^*, \ldots\ldots, v_{d,n}^*)^T$

where,

$u_{1,n}^* = \sqrt{n}(\omega_{1,n}^* - \omega_i)$ and$v_{1,n}^* = \sqrt{n}(\gamma_{1,n}^* - \gamma_i)$

We can rewrite the (1) and (2) as:

$$\begin{cases} \int_0^T \left(W_n(t) - \sum_{i=1}^{d} v_{i,n}\, G_i(t,\theta_n^*) \right) - \sum_{i=1}^{d} u_{i,n}\, F_i(t,\theta_n^*) G_\ell(\theta_n^*) = 0 \\ \int_0^T \left(W_n(t) - \sum_{i=1}^{d} v_{i,n}\, G_i(t,\theta_n^*) - \sum_{i=1}^{d} u_{i,n}\, F_i(t,\theta_n^*) \right) F_\ell(t,\theta_n^*) dt = 0 \end{cases}$$

or,

$$\begin{cases} \int_0^T W_n(t) G_i(t,\theta_n^*) dt = \sum_{i=1}^{d} v_{in} \int_0^T G_i\,(t,\theta_n^*) G_\ell(t,\theta_n^*) dt + \sum_{i=1}^{d} u_{i,n} \int_0^T F_i\,(t,\theta_n^*) G_e ll(t,\theta_n^*) dt = 0 \\ \int_0^T W_n(t) F_i(t,\theta_n^*) dt = \sum_{i=1}^{d} v_{i,n} \int_0^T G_i\,(t,\theta_n^*) F_\ell(t,\theta_n^*) dt + \sum_{i=1}^{d} u_{i,n} \int_0^T F_i\,(t,\theta_n^*) F_e ll(t,\theta_n^*) dt = 0 \end{cases}$$

Let us define the vectors:

$$Y_n = \left(Y_n^{(1)}, Y_n^{(2)} \right)^t Y_n^{(1)} = \begin{pmatrix} Y_{1,n}^{(1)} \\ \vdots \\ Y_{d,n}^{(1)} \end{pmatrix} Y_n^{(2)} = \begin{pmatrix} Y_{1,n}^{(1)} \\ \vdots \\ Y_{d,n}^{(2)} \end{pmatrix}$$

where,

$$Y_{\ell,n}^{(1)} = \int_0^T W_n(t) G_\ell(t, \theta_n^*) dt$$
$$Y_{\ell,n}^{(1)} = \int_0^T W_n(t) G_\ell(t, \theta_o) dt$$
$$Y_n^{(2)} = \int_0^T W_n(t) F(t, \theta_n^*) dt \;\; Y_\ell^{(2)} = \int_0^T W_n(t) F_\ell(t, \theta_o)$$

Thus,

$$Y_n^{(1)} = A(\theta_o) v_n^* + B(\theta_o) u_n^* + 0(1)$$
$$Y_n^{(2)} = B(\theta_o) v_n^* + C(\theta_o) u_n^* + 0(1)$$
$$D(\theta_o) \begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} = \begin{pmatrix} Y_n^{(1)} \\ Y_n^{(2)} \end{pmatrix} + 0(1)$$

By hypothesis B_1 the matrix $D(\theta_o)$ is invertible thus we can write $\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} = K(\theta_o) Y_n + 0(1)$ where $K(\theta_o) = D(\theta_o)^{-1}$.
Now we can finish the proof of the theorem.
We have:

$$n\, E_{\theta_o} (\theta_n^* - \theta_o)(\theta_n^* - \theta_o)^t = E_{\theta_o}[(K(\theta_o) Y_n)(K(\theta_o) Y_n)^t] + 0(1) = K(\theta_o) \left[E_{\theta_o} (Y_n Y_n^t) \right] K(\theta_o)^T + 0(1)$$

Introduce as well:

$$L_\ell(s) = \int_0^T G_\ell\,(t, \theta_o) dt \quad M_\ell(s) = \int_0^T F_\ell\,(t, \theta_o) dt$$

On the other hand:

$$\int_0^T W_n\,(t) G_\ell(t, \theta_o) dt = \int_0^T G_\ell\,(t, \theta_o) \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^T 1_{s<t}\,(dX_j(s) - S(\theta_o, s) ds dt$$
$$= \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^T \left(\begin{array}{l} \int_0^T G_\ell\,(t, \theta_o) 1_{s<t} dt)(dX_j(s) - S(\theta_o, s) ds \\ = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^T L_\ell\,(s) \pi_j(ds) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \psi_{\ell,j}^{(1)} \end{array} \right.$$

where,

$\pi_j(t) = X_j(t) - \Lambda(\theta_o, t)$ and by the similar way we obtain:

$$\int_0^T W_n\,(t) F_\ell(t, \theta_o) dt = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^T M_\ell\,(s) \pi_j(ds) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \psi_{\ell,j}^{(2)}$$

Using the properties of the stochastic integral with respect to the Poisson process, we can write:

$$E_{\theta_o} \left[X_j(t) - \Lambda(\theta_o, t) \right] \left[X_j(s) - \Lambda(\theta_o, s) \right] = \Lambda(\theta_o, t \wedge s)$$

and obtain, for example:

$$E_{\theta_o} \psi_{\ell,j}^{(1)} \psi_{\ell,j}^{(2)} = \int_0^T \int_0^T G_\ell\,(\theta_o, t) F_\ell(\theta_o, s) \Lambda(\theta_o, s) \Lambda(\theta_o, t \wedge s) ds dt$$

We note that the random vectors $\psi_j = (\psi_{i,j}^{(1)}, \dots, \psi_{2d,j}^{(1)})_{j=1,\dots,n}$ are independant and identically distributed with the following monents $E_{\theta_o}\psi_j = 0$ and:

$$E_{\theta_o}\psi_{\ell_{ij}}^{(p)}\psi_{\ell_{ij}}^{(q)} = \begin{cases} \int_0^T L_\ell(s)L_i(s)S(\theta_o,s)ds & \text{for } p=q=1 \\ \int_0^T L_\ell(s)M_i(s)S(\theta_o,s)ds & \text{for } p=q=2 \\ \int_0^T M_\ell(s)S(\theta_o,s)ds & \text{for } p=q=2 \end{cases}$$

Hence,

$$E_{\theta_o}\psi_{\ell_{ij}}^{(p)}\psi_{\ell_{ij}}^{(q)} = R_{\ell,i}^{p,q}(\theta_o) \ \ p,q = 1,2$$

where, the terms of the matrix $R(\theta_o)$ can be written as:

$$R_{\ell,i}^{(1,1)}(\theta_o) = \int_0^T \left[\int_t^T G_\ell(\theta_o,s)ds \int_t^T G_i(\theta_o,r)dr\right] S(\theta_o,t)dt$$

$$R_{\ell,i}^{(1,2)}(\theta_o) = \int_0^T \left[\int_t^T G_\ell(\theta_o,s)ds \int_t^T G_i(\theta_o,r)dr\right] S(\theta_o,t)dt$$

$$R_{\ell,i}^{(2,1)}(\theta_o) = \int_0^T \begin{bmatrix} \int_t^T F_\ell(\theta_o,s)ds \int_t^T G_\ell(\theta_o,r)drS(\theta_o,t)dt \\ \\ R_{\ell,i}^{(2,2)}(\theta_o) = \int_0^T \left[\int_t^T F_\ell(\theta_o,s)ds \int_t^T F_\ell(\theta_o,r)drS(\theta_o,t)dt\right. \end{bmatrix}$$

By applying the central limit theorem, we get $Y_n \Longrightarrow N(0, R(\theta_o))$.
As $\begin{pmatrix} u_n^* \\ v_n^* \end{pmatrix} = \sqrt{n}(\theta_n^* - \theta_o) = K(\theta_o)Y_n + 0(1)$, it follows that:

$$\sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow N(0, K(\theta_o)R(\theta_o)K^t(\theta_o) \text{ i.e.,}$$

$$\sqrt{n}(\theta_n^* - \theta_o) \Longrightarrow \mathcal{N}(0, D^{-1}(\theta_o)R(\theta_o)D^{-1}(\theta_o))$$

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