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Research Article

Nonparametric Testing for More DMRL Ordering

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Abstract: A test proposed for testing the null hypothesis that two life distributions are equal against the alternative that one is more Decreasing Mean Residual Life (DMRL) ordering. The asymptotic normality of the proposed test statistic was also established. The empirical size and empirical power of the proposed test are simulated for some specific families of distributions like beta and Weibull that are ordered with respect to more DMRL order. Finally, we apply our test to some real data sets.

Keywords: Asymptotic normality, empirical power, empirical size, kernel function, more DMRL

INTRODUCTION

Partial orderings of life distributions as more Decreasing Mean Residual Life (more DMRL) used in many systems such as biological, physical and mechanical systems to compare the aging properties of two arbitrary life distributions. We mean, an older system has a shorter remaining life time than a newer one (Hollander and Proschan, 1984).

Let X be a non-negative random variables with absolutely continuous distribution function F. It is said that F has Decreasing Mean Residual Life (DMRL) if $\mu_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ is decreasing in $x \geq 0$. If $\Im$ is the class of distribution function F on $[0,\infty]$, with $F(0)=0$. Let F, G $\in$ $\Im$ have $\mu_F(x)$, $\mu_G(x)$, then F is more decreasing mean residual life than G and write $F <_{DMRL} G$ if $\mu_F(F^{-1}(u))/\mu_G(G^{-1}(u))$ is non-increasing in $u \in [0,1]$ or $W_F^{-1} \text{o} W_G(u)$ is star-shaped in $u \in [0,1]$, where W_F and W_G are proper distribution functions on $[0,1]$. They are related to the scaled total time on test transform $W_F(u) = H_F^{-1}(u)$, where $H_F^{-1}(u) = F_e \text{o} F^{-1}(u)$ and $\overline{F}_e(u)$ is the equilibrium survival function.

Pandit and Gudaganavar (2009) and Pandit and Inginashetty (2012) discussed test the null hypothesis that F is equivalent to G against the alternative hypothesis that F is more IFR than G, Izad and Khaledi (2012) discussed the above problem against the alternative hypothesis that F is more IFRA than G also Hollander *et al*. (1986), Lim *et al*. (2005) and Pandit and Math (2012) discussed the above problem against the alternative hypothesis that F is more NBU than G.

In material and methods section we propose a new test for testing that the null hypothesis that F is equivalent to G against the alternative hypothesis that F is more DMRL than G ($F <_{DMRL} G$).

The Proposed two-sample more DMRL ordering test: Let $X_1,..., X_n$ and $Y_1,..., Y_m$ denote two random samples from absolutely continuous distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$ and density functions f and g, respectively.

Since $F <_{DMRL} G \Leftrightarrow \frac{\mu_F(F^{-1}(u))}{\mu_G(G^{-1}(u))}$ is non-increasing in $u \in [0,1] \Leftrightarrow \Delta_{F,G} \geq 0$, $u \in [0,1]$, where,

$$\Delta_{F,G} = f(F^{-1}(u))\nu(F^{-1}(u)) - g(G^{-1}(u))\nu(G^{-1}(u))$$

and $\nu(F^{-1}(u)) = \int_{F^{-1}(u)}^\infty \overline{F}(x)dx$.

For a distribution H with density h, define:

$$\Delta(H) = \int_0^\infty \nu(x)h(x)dH(x)$$

and:

$$\delta(F,G) = \int_0^1 \Delta_{F,G}(u)du = \Delta(F) - \Delta(G)$$

We use $\delta(F,G)$ to test the null hypothesis $H_0 : F =_{DMRL} G$ against the alternative hypothesis $H_1 : F \leq_{DMRL} G$ and $F \neq_{DMRL} G$. As introduced by Ahmad (2000), the function $\Delta(F)$ can be estimated by:

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$$\hat{f}_n(x) = \frac{1}{na_n}\sum_{i=1}^{n} k(\frac{x - X_i}{a_n})$$

where,

$$\hat{g}_m(x) = \frac{1}{mb_n}\sum_{i=1}^{m} k(\frac{x - Y_i}{b_m}) \qquad (1)$$

and:

$$\hat{v}_n(x) = \frac{1}{n}\sum_{j=0}^{n}(X_j - x)I(X_j > x)$$

And F_n is empirical distribution of *F*. The function *k* in (1) is a known symmetric and bounded probability density function with mean 0 and finite variance σ_k^2 and $\{a_n\}$ is a equence of positive real numbers such that $a_n \to 0$ as $n \to \infty$. The density function *K* is known as kernel function and sequence $\{a_n\}$ is known as bandwidth (Silverman, 1986).

Also, let b_m be a sequence of positive real numbers such that $b_m \to 0$ as $m \to \infty$, then similar to $\hat{\Delta}(F)$, $\hat{\Delta}(G)$ can be estimated by:

$$\hat{\Delta}(G) = \int_0^{\infty} \hat{v}_m(x)\hat{g}_m(x)dG_m(x)$$

where,

$$\hat{g}_m(x) = \frac{1}{b_m}\sum_{i=1}^{m} k(\frac{x - Y_i}{b_m}), \qquad (2)$$

and:

$$\hat{v}_m(x) = \frac{1}{m}\sum_{j=0}^{m}(Y_j - x)I(Y_j > x)$$

And G_m is empirical distribution of *G*. Now, the test statistic:

$$\hat{\delta}(F,G) = \hat{\Delta}(F) - \hat{\Delta}(G) = \frac{1}{n}\sum_{i=1}^{n}\hat{v}_n(X_i)\hat{f}_n(X_i) - \frac{1}{m}\sum_{i=1}^{m}\hat{v}_m(Y_i)\hat{g}_m(Y_i)$$

$$= \frac{1}{n^3 a_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(X_j - X_i)I(X_j > X_i)k(\frac{X_i - X_l}{a_n})$$

$$-\frac{1}{m^3 b_m}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}(Y_j - Y_i)I(Y_j > Y_i)k(\frac{Y_i - Y_l}{b_m})$$

Can be used to test $H_0 : F =_{DMRL} G$ against $H_1 : F <_{DMRL} G$ and $F \neq_{DMRL} G$. We need the following theorem due to Ahmed (2000) to obtain the asymptotic distribution of test statistic.

Theorem 1: If $na_n \to \infty$ and $na_n^4 \to 0$, If *f* has bounded first and second derivative and if $lim_n V(\hat{\Delta}_F) > 0$, the $\sqrt{n}(\hat{\Delta} \to \Delta_F)$ is asymptotically with mean 0 and variance σ_F^2 given by:

$$\sigma_f^2 = V[X_1\int_0^{X_1} f(x)dF(x) - \int_0^{X_1} xf(x)dF(x)] + 2[f(X_1)\int_{X_1}^{\infty} xdF(x) - X_1 f(X_1)\overline{F}(X_1)] \quad (3)$$

Theorem 2: Let a_n and b_m be two sequences of positive real numbers such that:

- $a_n \to 0$, $na_n \to \infty$ and $na_n^4 \to 0$ as $n \to \infty$.
- $b_m \to 0$, $mb_m \to \infty$ and $mb_m^4 \to 0$ as $m \to \infty$.
- For all $m, n \in N$ and some constant $c \in (0, \frac{1}{2}]$ we have that:

  $0 < c \leq \frac{n}{N} \leq 1 - c < 1$, where $N = n + m$

If *f* and *g* have bounded first and second derivatives such that for all combination of $p = 1,2$, $q = 1,2,3$ and $r = 2$.

$$\int_0^{\infty} x^p (f(x))^q f^{(r)}(x)dx < \infty$$

$$\int_0^{\infty} x^p (g(x))^q g^{(r)}(x)dx < \infty$$

Then as $min(n,m) \to \infty$, $\sqrt{n}(\hat{\delta}(F,G) - \delta(F,G))$ is asymptotically normal with mean 0 and variance $\sigma_{f,g}^2$ given by:

$$\sigma_{f,g}^2 = \frac{N}{n}\sigma_f^2 + \frac{N}{m}\sigma_g^2$$

where, σ_f^2 is given in 3 and σ_g^2 is defined similarly.

Proof: The proof of the required result is based on Theorem (1) and the fact that generally convergence in the distribution is closed under the convolution.
We use the following lemma to show that in Theorem (3) that $\hat{\sigma}_{f,g}^2$ given by:

$$\hat{\sigma}_{f,g}^2 = \frac{N}{n}[\int_0^{\infty} \hat{v}_n^2(x)\hat{f}_n^2(x)dF_n(x) - \hat{\Delta}^2(F)] + \frac{N}{m}[\int_0^{\infty} \hat{v}_m^2(x)\hat{g}_m^2(x)dG_m(x) - \hat{\Delta}^2(G)]$$

$$= \frac{N}{n^6 a_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}(X_j - X_i)(X_r - X_i)K(\frac{X_i - X_l}{a_n})K(\frac{X_i - X_s}{a_n})$$

$$I(X_j > X_i)I(X_r > X_i) - \frac{N\hat{\Delta}^2(F)}{n} + \frac{N}{m^6 b_m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\sum_{r=1}^{m}\sum_{s=1}^{m}(Y_j - Y_i)$$

$$(Y_r - Y_i)K(\frac{Y_i - Y_l}{b_m})K(\frac{Y_i - Y_s}{b_m})I(Y_j > Y_i)I(Y_r > Y_i) - \frac{N\hat{\Delta}^2(G)}{m}$$

Is the consistent estimator of $\sigma_{f,g}^2$.

Lemma 1: Let a_n be a sequence of positive real numbers such that $na_n^2 \to \infty$ as $n \to \infty$ and let f be uniformly continuous. Then:

- $\hat{\Delta}(F) \to^p \Delta(F)$, if $\int \nu^p(x) f^q(x) dF(x)$ is finite for $(p,q) \in (1,0),(1,1)$ and
- $\int \hat{\nu}_n^2(x) \hat{f}_n^2(x) dF_n(x) \to^p \int \nu^2(x) f^2(x) dF(x)$ is finite for $(p,q) \in (2,0),(2,1),(2,2)$.

Proof: (i) Using triangular inequality, we have that:

$$\begin{aligned} &| \int \hat{\nu}_n(x) \hat{f}_n(x) dF_n(x) - \int \nu(x) f(x) dF(x) | \le \\ &\qquad | \int \hat{\nu}_n(x) \hat{f}_n(x) dF_n(x) - \int \nu(x) f(x) dF_n(x) | \\ &\qquad + | \int \nu(x) f(x) dF_n(x) - \int \nu(x) f(x) dF(x) | \\ &\qquad = A_n + B_n \end{aligned} \tag{4}$$

Since $\Delta(F) < \infty$, the weak law of large numbers implies that $B_n \to 0$ in probability as:

$n \to \infty$

Next, it is clear that:

$$A_n \le sup \mid \hat{f}_n(x) - f(x) \mid [\int \hat{\nu}_n(x) dF_n(x) + \int \nu(x) dF(x)]$$

Using the assumptions that f is uniformly continuous and $na_n^2 \to \infty$ as $n \to \infty$, it follows from Theorem 3A of Parzen (1962) that $sup \mid \hat{f}_n(x) - f(x) \mid$ converges to 0 in probability. On the other hand, by the weak law of large numbers we have that:

$$\int \hat{\nu}_n(x) dF_n(x) \to p \int \nu(x) dF(x)$$

Hence, $A_n \to 0$ in probability as $n \to \infty$ and proof of part (i) is completed.

- In this case we also have that:

$$\begin{aligned} &| \int \hat{\nu}_n^2(x) \hat{f}_n^2(x) dF_n(x) - \int \nu^2(x) f^2(x) dF(x) | \le | \int \{\hat{\nu}_n^2(x) \hat{f}_n^2(x) - \nu^2(x) f^2(x)\} dF_n(x) | \\ &+ | \int \nu^2(x) f^2(x) \{dF_n(x) - dF(x)\} | = A_n + B_n \end{aligned}$$

By the weak law of large numbers, B_n converges to 0 in probability as $n \to \infty$ since by assumption $\int \nu^2(x) f^2(x) dF(x) < \infty$. It is easy to see that:

$$A_n \le sup \mid \hat{f}_n(x) - f(x) \mid [\int \hat{\nu}_n^2(x) \hat{f}_n(x) dF_n(x) + \int \nu^2(x) f(x) dF_n(x)] \tag{5}$$

Now, by assumption that $na_n^2 \to \infty$ as $n \to \infty$, the first term on the right hand side of (5) converges in probability to 0 by Theorem 3A in Parzen (1962). The second term in the bract, on the right hand side of 5, by the weak law of large numbers, converges to $\int \nu^2(x) f(x) dF(x)$ as $n \to \infty$, since $\int \nu^2(x) f(x) dF(x) < \infty$. On the other hand, since $\int \nu^2(x) dF(x) < \infty$, replacing $\nu(x)$ with $\nu^2(x)$ in (4) we obtain that:

$$\int \hat{\nu}_n^2(x) \hat{f}_n(x) dF_n(x) \to \int \nu^2(x) f(x) dF(x)$$

when,

$na_n^2 \to \infty$ as $n \to \infty$

Combining these observations, the required result of part (ii) follows.

Theorem 3: Let $\{a_n\}$ and $\{b_m\}$ be two sequences, of positive real numbers such that:

$$na_n^2 \to \infty \text{ as } n \to \infty$$
$$nb_m^2 \to \infty \text{ as } m \to \infty$$

and:

Let $\int \nu^p(x) f^q(x) dF(x)$ be finite for $(p,q) \in \{(1,0), (1,1), (2,0), (2,1), (2,2)\}$ and f be uniformly continuous. Then $\hat{\sigma}^2_{f,g} \to_p \sigma^2_{f,g}$ as $min(n,m) \to \infty$.

Proof: Since:

$$\hat{\sigma}_f^2 = \int \hat{\nu}_n^2(x) \hat{f}_n^2(x) dF_n - \hat{\Delta}^2(F)$$

$$\hat{\sigma}_g^2 = \int \hat{\nu}_m^2(x) \hat{g}_m^2(x) dG_m - \hat{\Delta}^2(G)$$

It follows from 1 that $\hat{\sigma}_f^2 \to_p \sigma_f^2$ and $\hat{\sigma}_g^2 \to_p \sigma_g^2$ as $min(n,m) \to \infty$, which in turn imply that $\hat{\sigma}^2_{f,g} \to_p \sigma^2_{f,g}$ as $min(n,m) \to \infty$.

With the results of Theorem 2 and 3, it follows from Slutsky's theorem that $\sqrt{N}\hat{\delta}(F,G)/\hat{\sigma}_{f,g}$ is asymptotically standard normal and therefore we reject H_0 in favor of H_1 at level α if $\sqrt{N}\hat{\delta}(F,G)/\hat{\sigma}_{f,g} > z_\alpha$, where, z_α is $(1-\alpha)$-quantile of standard normal distribution.

Empirical size and empirical power: We will use Weibull and beta families of distributions for simulation study to justify the power of our proposed test and the accuracy of the standard normal as a limit distribution of the test statistic under H_0 (empirical size). The choice of Weibull family and beta family because are ordered according to more IFRA ordering (Chandra and Singpurwalla, 1981; Van Zwet, 1970; Izadi and Khaledi, 2012) then it is also ordered according to more DMRL.

The algorithm of calculate the empirical size:

- We take the following distributions as a kernel functions
  - Standard normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}} Exp[-\frac{1}{2}x^2], -\infty < x < \infty$$

  - (Uniform distribution: $f(x) = \frac{1}{2}, -1 < x < 1$

Because are bounded and symmetric distributions about 0 and have finite variance.

- The samples size are given by $n = m = 100, 200$
- The significance level α is given by $\alpha = 0.05, 0.1$
- For each significance level, preform 10^4 simulation run and compute the empirical size

In Table 1 and 2, we compute the empirical size of the proposed test according to the above algorithm. The entries in these table show that the empirical size is increasing in m and n. Also the proposed test is powerful because the empirical size is similar to significant level α.

The algorithm of calculate of the empirical power:

- We take the following distributions as a kernel functions
  - Standard normal distribution
  - $f(x) = \frac{1}{\sqrt{2\pi}} Exp[-\frac{1}{2}x^2], -\infty < x < \infty$
  - Uniform distribution $f(x) = \frac{1}{2}, -1 < x < 1$

Because are bounded and symmetric distributions about 0 and have finite variance.

- The samples size are given by $n = m = 10, 30, 40, 50, 100$.
- The significance level α is given by $\alpha = 0.05, 0.$
- For each significance level, preform 10^4 simulation run and compute the empirical size and obtain the number of rejections of H_0. Dividing the number of rejections by 10^4, the empirical power are computed.

In Table 3 and 4, we obtain the empirical powers of the proposed test according to the above algorithm. We observe that the power of test is close to 1 when $n = m = 20$ for beta distribution. Also, when $n = m = 30$ the empirical power for Weibull distribution is large even for the cases when $n = m = 20$, $n = m = 40$, $n = m = 50$ and $n = m = 100$.

Numerical example: Table 5 shows two sets of life test data corresponding to snubber designs of a toaster component taken from Table (8.3.1) of Nelson (1982).

Ahmad (2000) showed that under the null hypothesis $H_0 : F =_{DMRL} E$ (F is an exponential

Table 1: The empirical size of the proposed test when the kernel function is standard normal

Distribution		m = n = 100		n = m = 200	
F	G	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.1$
B (0.5, 1)	B (0.5, 1)	0.038	0.023	0.177	0.282
B (1.5, 1)	B (1.5, 1)	0.036	0.038	0.131	0.227
W (5, 1)	W (5, 1)	0.027	0.021	0.091	0.179
W(6,1)	W(6,1)	0.021	0.027	0.144	0.233

Table 2: The empirical size of the proposed test when the kernel function is uniform distribution (-1, 1)

Distribution		m = n = 100		n = m = 200	
F	G	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.1$
B (0.5, 1)	B (0.5, 1)	0.058	0.064	0.120	0.130
B (1.5, 1)	B (1.5, 1)	0.024	0.074	0.109	0.132
W (5, 1)	W (5, 1)	0.031	0.048	0.122	0.135
W (6, 1)	W (6, 1)	0.029	0.048	0.094	0.069

Table 3: The empirical power of the proposed test when the kernel function is standard normal

Distribution		Sample sizes				
F	G	m = n = 20	m = n = 30	n = m = 40	n = m = 50	n = m = 100
B (0.5,1)	B (0.5, 1)	0.998	0.500	0.914	0.500	0.500
B (1.5, 1)	B (1.5, 1)	0.760	0.741	0.589	0.527	0.564
W (5, 1)	W (5, 1)	0.500	0.500	0.500	0.500	0.508
W (6, 1)	W (6, 1)	0.500	0.500	0.500	0.500	0.500

Table 4: The empirical size of the proposed test when the kernel function is uniform distribution (-1, 1)

Distribution		Sample sizes				
F	G	m = n = 20	m = n = 30	n = m = 40	n = m = 50	n = m = 100
B (0.5, 1)	B (0.5, 1)	1	0.547	0.504	0.696	0.505
B (1.5, 1)	B (1.5, 1)	1	0.500	0.584	0.500	0.500
W (5, 1)	W (5, 1)	0.500	0.500	0.500	0.500	0.507
W (6, 1)	W (6, 1)	0.500	0.992	0.500	0.543	0.500

Table 5: Life test of two different snubber designs

Old design	90- 90- 260- 410- 410- 485- 508- 631- 631- 631- 635- 658- 731- 739- 790- 855- 980- 980
New design	47- 73- 145- 311- 490- 571- 575- 608- 575- 608- 608- 630- 670- 670- 838- 964- 964- 1198

distribution), $\Delta(F) = \frac{1}{3}$ and $\sigma_f^2 = \frac{1}{3}$. Thus H_0 is rejected in favor of alternative hypothesis $H_1 : F <_{DMRL} E$ (F is not exponential distribution) at significance level α if $H = \sqrt{3n}(\hat{\Delta}(F) - \frac{1}{3}) > Z_\alpha$. Now, to compute the statistic H for these data sets, select standard normal density function as the kernel function. For the old design data, $H = 6.8227$ with $P-value = 0.000$ and for the new design data 8.79876 with $P-value = 0.000$. Therefore, the test also confirms that the data-sets come from some DMRL populations.

Now, to compute the statistic H for these data sets, select standard normal density function as the kernel function and let $a_n = n^{\frac{-2}{5}}$. For the old design data, $H = 6.8227$ with $P-value = 0.000$ and for the new design data 8.79876 with $P-value = 0.000$.

Therefore, the test also confirms that the data-sets come from some DMRL populations. To compare these two data sets with respect to more DMRL order, assume that F is distribution function of the old design data and G is distribution function of new design data. Under these setting $\sqrt{N}\hat{\delta}(F,G)/\sigma_{f,g} = 5.92917$ and $P-value = 0.000$.

Thus, we conclude that the old and new designs are ordered according to more DMRL ordering.

CONCLUSION

In this study, we proposed a new test that two unknown distributions are identical against the other alternative that one is more DMRL than the other. We proof that the proposed test is asymptotically normal and consistent. Also presents two algorithm to compute the empirical size and empirical power. The results of computing the empirical size and empirical power are the empirical size is increasing as *n* and *m* increasing. The proposed test is powerful because the empirical size is similar to significant level *α*.

REFERENCES

Ahmad, I.A., 2000. Testing exponentiality against positive ageing using kernel methods. Sankhya Ser. A, 62: 244-257.

Chandra, M. and N.D. Singpurwalla, 1981. Relationships between some notions which are commontoreliability theory and economics. Math. Oper. Res., 6: 113-121.

Hollander, M. and F. Proschan, 1984. Nonparametric Concepts and Methods in Reliability. In: Krishnaiah, P.R. and P.K. Sen (Eds.), Handbook of Statistics. North Holland, New York, 4: 613-655.

Hollander, R.M., D.H. Park and F. Proschan, 1986. Testing whether F is "more NBU" than is G. Microelectron. Reliab., 26(1): 39-44.

Izadi, M. and B. Khaledi, 2012. On testing more IFRA ordering. J. Stat. Plan. Infer., 142: 840-847

Lim, J.H., D.K. Kim and D.H. Park, 2005. Tests for detecting more NBU-ness at specific age: Australian and Newziland. J. Stat., 47(3): 329-337.

Nelson, W., 1982. Applied Life Data Analysis. Wiley, NewYork.

Pandit, P.V. and N.V. Gudaganavar, 2009. On two-sample test for detecting differences in the IFR property of life distributions. Afr. J. Math. Comput. Sci. Res., 2(2): 25-29.

Pandit, P.V. and C.M. Math, 2012. TEST procedures for detecting more NBU-ness property of life distributions. Adv. Dev. Technol. Int., 1: 72-82.

Pandit, P.V. and S. Inginashetty, 2012. A note on detecting more IFR-ness property of life distributions. Open J. Stat., 2: 443-446.

Parzen, E., 1962. On the estimation of aprobability density function and mode. Ann. Math. Stat., 33: 1065-1076.

Silverman, B.W., 1986. Density Estimation for Statistics and Data Analysis. Chapman and Hall, London.

Van Zwet, W.R., 1970. Convex Transformations of Random Variables. 2nd Edn., Mathematische Centrum, Amsterdam.