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Research Article

An Efficient Method for Second Order Boundary Value Problems

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Abstract: In this study, we apply the homotopy perturbation method for solving the second-order boundary value problems by reformulating them as an equivalent system of integral equations. This equivalent formulation is obtained by using a suitable transformation. The analytical results of the integral equations have been obtained in terms of convergent series with easily computable components. The method is tested for solving linear second-order boundary value problems. The analysis is accompanied by several examples that demonstrate the comparison and shows the pertinent features of the homotopy perturbation technique.

Keywords: Boundary value problems, homotopy perturbation method, integral equation

INTRODUCTION

In this study, we consider the general second-order boundary value problems of the type:

$$y^{(2)}(x) = f(x, y, y') \tag{1}$$

With boundary conditions:

$$y(a) = \alpha_1, y(b) = \beta_1 \tag{2}$$

where, f is continuous function on [a, b] and the parameters α_1 and β_1 are real constants.

The second-order boundary value problems generally arise in the mathematical modeling of viscoelastic flows. The second-order boundary value problems were investigated by Fang *et al*. (2002) by Finite difference, finite element and finite volume methods and considered by Caglar *et al*. (2006) by means of B-spline interpolation method. Moreover, the Cubic Spline Methods were used by Rashidinia *et al*. (2008) and the obtained results produced improvements over other works. However, the performance of the approaches used so far is well known in that it provides the solution at grid points only. We should point out that these approaches which were provided to solve this type of problems require a large amount of computational effort. The present work is motivated by the desire to obtain analytical and numerical solutions to boundary value problems for higher-order differential equations. The homotopy perturbation method has been shown to solve effectively, easily and accurately a large class of linear, non-linear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. This method provides the approximate solutions without discretization and the computation of the Adomian polynomials and the coefficients of cubic spline and B-spline functions. Unlike analytical perturbation methods, the homotopy perturbation method does not depend on a small parameter which is difficult to find. Several examples are given to illustrate the performance of this method.

HOMOTOPY PERTURBATION METHOD

To delineate the Homotopy Perturbation Method (HPM), the following nonlinear differential equation was considered by He (1999):

$$A(u) = f(r), r \in \Omega \tag{3}$$

With boundary conditions:

$$B(u, \partial u/\partial n) = 0, r \in \Gamma \tag{4}$$

where,
A = A general differential operator
B = A boundary operator
$f(r)$ = A known analytic function
Γ = The boundary of the domain.

The operator A can, generally speaking, be divided into two parts M and N, where M is linear, while N is non-linear. Therefore, (3) can be rewritten as follows:

$$M(u) + N(u) - f(r) = 0 \tag{5}$$

A homotopy $v(r, \varepsilon): \Omega \times [0,1] \to R$ was constructed by He (1999, 2000) which satisfies:

$$H(v, \varepsilon) = (1 - \varepsilon)[M(v) - M(u_0)]$$

$$+\varepsilon[A(v)-f(r)]=0 \tag{6}$$

Or:

$$(v,\varepsilon)=M(v)-M(u_0)+\varepsilon M(u_0)+\varepsilon[N(v)-f(r)]=0$$

where, $r\in\Omega$ and $\varepsilon\in[0,1]$ is an imbedding parameter and u_0 is an initial approximation of (3). Clearly, we have:

$$H(v,0)=M(v)-M(u_0)=0$$

and:

$$H(v,1)=A(v)-f(r)=0 \tag{7}$$

And changing the variation of ε from 0 to 1 is the same as changing $H(v,\varepsilon)$ from $M(v)-M(u_0)$ to $A(v)-f(r)$. In topology, this is called deformation, while $M(v)-M(u_0)$ and $A(v)-f(r)$ are called homotopic. Owing to the fact that $0\le\varepsilon\le 1$ can be considered as a small parameter, by applying the classical perturbation technique, we can assume that the solution of (7) can be expressed as a series in ε, as follows:

$$v=v_0+\varepsilon v_1+\varepsilon^2 v_2+\varepsilon^3 v_3+\cdots \tag{8}$$

when $\varepsilon\to 1$, Eq. (6) and (7) correspond to Eq. (5) and (9) becomes the approximate solution of Eq. (5), i.e.,:

$$u(x)=\lim_{\varepsilon\to 1} v=v_0+v_1+v_2+v_3+\cdots$$

The combination of the perturbation method and the homotopy method is known as HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages.

NUMERICAL RESULTS

We first show that the second-order boundary value problems may be reformulated as a system of integral equations. In this section, the method is tested on the following second-order boundary value problems. For the sake of comparison, we consider the same examples as that of references.

Example 1: Consider the second-order boundary value problem:

$$y''(x)-y'(x)=-e^{x-1}-1, 0<x<1 \tag{9}$$

Subject to the boundary conditions:

$$y(0)=0\,, y(1)=0 \tag{10}$$

The exact solution of this problem is:

$$y(x)=x(1-e^{x-1}) \tag{11}$$

We define:

$$\frac{dy}{dx}=z(x), \frac{dz}{dx}=y'(x)-e^{x-1}-1$$

Then we can rewrite the second order boundary value problem as the system of following differential equations:

$$y(x)=y(0)+\int_0^x z(t)dt$$
$$z(x)=z(0)+\int_0^x (y'(t)-e^{t-1}-1)dt$$

Using the homotopy perturbation method, we obtain:

$$y_0+py_1+p^2y_2+\cdots=0+p\int_0^x (z_0+pz_1+p^2z_2+\cdots)dt$$
$$z_0+pz_1+p^2z_2+\cdots=a+p\int_0^x (z_0+pz_1+p^2z_2+\cdots(-e^{t-1}-1))dt$$

Comparing the coefficients of like powers of p, adding all terms, the solution is given as:

$$y(x)=\frac{10}{e}-10e^{x-1}+ax+\frac{10x}{e}-\frac{x^2}{2}+\frac{ax^2}{2}+\frac{9x^2}{2e}-\frac{x^3}{6}+\frac{ax^3}{6}+\frac{4x^3}{3e}-\frac{x^4}{24}+\frac{ax^4}{24}+\frac{7x^4}{24e}-\frac{x^5}{120}+\frac{ax^5}{120}+\frac{x^5}{20e}-\frac{x^6}{720}+\frac{ax^6}{720}+\frac{x^6}{144e}-\frac{x^7}{5040}+\frac{ax^7}{5040}$$

Using the boundary conditions at $x=1$, we have a = 0.632121.

Table 1 shows the comparison between exact solution and the numerical solution obtained using the proposed homotopy perturbation method. The maximum absolute error obtained by the proposed method is compared with that of obtained by Fang *et al*. (2002), Caglar *et al*. (2006), Rashidinia *et al*. (2008) and Chang *et al*. (2011) in Table 2.

Table 1: Absolute errors for the example 1

x value	Exact Solution	Present Study	Error
0.0	0.00000000000	0.00000000000	0.000000
0.1	0.05934303403	0.05934303392	1.1×10^{-10}
0.2	0.11013420718	0.11013420695	2.2×10^{-10}
0.3	0.15102440886	0.15102440851	3.5×10^{-10}
0.4	0.18047534556	0.18047534507	5.0×10^{-10}
0.5	0.19673467014	0.19673466949	6.5×10^{-10}
0.6	0.19780797238	0.19780797155	8.2×10^{-10}
0.7	0.18142724552	0.18142724452	1.0×10^{-9}
0.8	0.14501539754	0.14501539641	1.12×10^{-9}
0.9	0.08564632377	0.08564632278	9.9×10^{-10}
1.0	0.00000000000	0.00000000000	0.000000

Table 2: Maximum absolute errors for example 1

References	Results
Fang *et al*. (2002)	8.24×10^{-5}
Caglar *et al*. (2006)	2.9×10^{-6}
Rashidinia *et al*. (2008)	1.88×10^{-3}
Chang *et al*. (2011)	0.0004
Present Study	1.12×10^{-9}

Example 2: Consider the second-order boundary value problem:

$$y''(x) = y(x), 0 < x < 1 \tag{12}$$

With boundary conditions:

$$y(0) = 0, y(1) = \sinh(1)$$

The exact solution of this problem is:

$$y(x) = sinhx \tag{13}$$

We define:

$$\frac{dy}{dx} = z(x),\ \frac{dz}{dx} = y(x)$$

Then we can rewrite the second order boundary value problem as the system of following differential equations:

$$y(x) = y(0) + \int_0^x z(t)dt$$
$$z(x) = z(0) + \int_0^x y(t)dt$$

Using the homotopy perturbation method, we obtain:

$$y_0 + py_1 + p^2y_2 + \cdots = 0 + p\int_0^x (z_0 + pz_1 + p^2z_2 + \cdots)dt$$
$$z_0 + pz_1 + p^2z_2 + \cdots = a + p\int_0^x (y_0 + py_1 + p^2y_2 + \cdots)dt$$

Comparing the coefficients of like powers of p, adding all terms.
The solution is given as:

$$y(x) = ax + \frac{ax^3}{6} + \frac{ax^5}{120} + \frac{ax^7}{5040} + \frac{ax^9}{362880} + \frac{ax^{11}}{39916800} + \frac{ax^{13}}{6227020800}$$

Using the boundary conditions at $x = 1$, we have $a = 1.000000$.

Table 3 shows the comparison between exact solution and the numerical solution obtained using the proposed homotopy perturbation method. The maximum absolute error obtained by the proposed method is compared with that of obtained by Hamid *et al.* (2010, 2011) in Table 4.

Table 3: Absolute errors for the example 2

x value	Exact Solution	Present Study	Error
0.0	0.00000000000	0.00000000000	0.0000000
0.1	0.10016675002	0.10016675002	6.5×10^{-14}
0.2	0.20133600254	0.20133600254	1.32×10^{-13}
0.3	0.30452029345	0.30452029345	1.99×10^{-13}
0.4	0.41075232580	0.41075232580	2.68×10^{-13}
0.5	0.52109530549	0.52109530549	3.40×10^{-13}
0.6	0.63665358215	0.63665358215	4.15×10^{-13}
0.7	0.75858370184	0.75858370184	4.92×10^{-13}
0.8	0.88810598219	0.88810598219	5.53×10^{-13}
0.9	1.02651672571	1.02651672571	5.13×10^{-13}
1.0	1.17520119364	1.17520119364	0.0000000

Table 4: Maximum absolute errors for example 2

References	Results
Hamid *et al*. (2010)	2.1178×10^{-4}
Hamid *et al*. (2011)	6.6718×10^{-7}
Present Study	5.53×10^{-13}

Example 3: Consider the second-order boundary value problem:

$$y''(x) = \pi^2 y(x) - 2\pi^2 \sin(\pi x)\,, 0 < x < 1 \tag{14}$$

With boundary conditions:

$$y(0) = 0\,, y(1) = 0$$

The exact solution of this problem is:

$$y(x) = \sin(\pi x) \tag{15}$$

We define:

$$\frac{dy}{dx} = z(x)$$
$$\frac{dz}{dx} = \pi^2 y(x) - 2\pi^2 \sin(\pi x)$$

Then we can rewrite the second order boundary value problem as the system of following differential equations:

$$y(x) = y(0) + \int_0^x z(t)dt$$
$$z(x) = z(0) + \int_0^x (\pi^2 y(t) - 2\pi^2 \sin(\pi t))\,dt$$

Using the homotopy perturbation method, we obtain:

$$y_0 + py_1 + p^2y_2 + \cdots = 0 + p\int_0^x (z_0 + pz_1 + p^2z_2 + \cdots)dt$$
$$z_0 + pz_1 + p^2z_2 + \cdots = a + p\int_0^x (\pi^2(y_0 + py_1 + p^2y_2 + \cdots) - 2\pi^2 \sin(\pi t))\,dt$$

Comparing the coefficients of like powers of p, adding all terms, the solution is given as:

Table 5: Absolute errors for the example 3

x value	Exact solution	Present study	Error
0.0	0.00000000000	0.00000000000	0.000000
0.1	0.30901699437	0.30901699375	6.2×10^{-10}
0.2	0.58778525229	0.58778525099	1.30×10^{-9}
0.3	0.80901699437	0.80901699226	2.11×10^{-9}
0.4	0.95105651630	0.95105651316	3.13×10^{-9}
0.5	1.00000000000	0.99999999553	4.47×10^{-9}
0.6	0.95105651630	0.95105651005	6.24×10^{-9}
0.7	0.80901699437	0.80901698576	8.62×10^{-9}
0.8	0.58778525229	0.58778524071	1.16×10^{-9}
0.9	0.30901699437	0.30901698107	1.33×10^{-9}
1.0	0.00000000000	0.00000000000	0.000000

Table 6: Maximum absolute errors for example 3

References	Results
Hamid *et al*. (2010)	2.0373×10^{-5}
Hamid *et al*. (2011)	5.1503×10^{-6}
Present study	8.62×10^{-9}

$$y(x) = ax + \frac{a\pi^2x^3}{6} - \frac{\pi^3x^3}{3} + \frac{a\pi^4x^5}{120} + \frac{a\pi^6x^7}{5040} - \frac{\pi^7x^7}{2520} + \frac{a\pi^8x^9}{362880} + \frac{a\pi^{10}x^{11}}{39916800} - \frac{\pi^{11}x^{11}}{19958400}$$

Using the boundary conditions at $x = 1$, we have $a = 3.14159$.

Table 5 shows the comparison between exact solution and the numerical solution obtained using the proposed homotopy perturbation method. The maximum absolute error obtained by the proposed method is compared with that of obtained by Hamid *et al*. (2010, 2011) in Table 6.

Example 4: Consider the second-order boundary value problem:

$$y''(x) = 4y(x), 0 < x < 1 \tag{16}$$

With boundary conditions:

$$y(0) = 1.1752,\ y(1) = 10.0179 \tag{17}$$

The exact solution of this problem is:

$$y(x) = \sinh(2x + 1)$$

We define:

$$\frac{dy}{dx} = z(x), \qquad \frac{dz}{dx} = 4y(x)$$

Then we can rewrite the second order boundary value problem as the system of following differential equations:

$$y(x) = y(0) + \int_0^x z(t)dt$$
$$z(x) = z(0) + \int_0^x (4y(t))dt$$

Table 7: Absolute errors for the example 4

x value	Exact solution	Present study	Error
0.0	1.175201194	1.175200000	1.194×10^{-6}
0.1	1.509461355	1.509461779	4.238×10^{-7}
0.2	1.904301501	1.904303559	2.058×10^{-6}
0.3	2.375567953	2.375571728	3.775×10^{-6}
0.4	2.942174288	2.942179932	5.644×10^{-6}
0.5	3.626860408	3.626868146	7.739×10^{-6}
0.6	4.457105171	4.457115315	0.00001014
0.7	5.466229213	5.466242170	0.00001295
0.8	6.694732228	6.694748518	0.00001629
0.9	8.191918354	8.191938630	0.00002027
1.0	10.01787492	10.01790000	0.00002507

Table 8: Maximum absolute errors for example 4

References	Results
Kashem (2009) Kashem (Rashidinia *et al*., 2008)	0.0045
Present study	0.0000250726

Using the homotopy perturbation method, we obtain:

$$y_0 + py_1 + p^2y_2 + \cdots = 1.1752 + p\int_0^x (z_0 + pz_1 + p^2z_2 + \cdots)dt$$
$$z_0 + pz_1 + p^2z_2 + \cdots = a + p\int_0^x \big(4(y_0 + py_1 + p^2y_2 + \cdots)\big)dt$$

Comparing the coefficients of like powers of p, adding all terms, the solution is given as:

$$y(x) = 1.1752 + ax + 2.3504x^2 + \frac{2ax^3}{3} + 0.78x^4 + \frac{2ax^5}{15} + 0.10x^6 + \frac{4ax^7}{315} + 0.007x^8 + \frac{2ax^9}{2835} + 0.0003x^{10} + \frac{4ax^{11}}{155925} + 0.00001x^{12} + \frac{4ax^{13}}{6081075}$$

Using the boundary conditions at $x = 1$, we have $a = 3.08618$.

Table 7 shows the comparison between exact solution and the numerical solution obtained using the proposed homotopy perturbation method. The maximum absolute error obtained by the proposed method is compared with that of obtained by Kashem (2009) in Table 8.

CONCLUSION

In this study, the homotopy perturbation method has employed to solve second-order boundary value problems. We then have conducted a comparative study between He's homotopy perturbation method and the traditional methods, i.e., Cubic Spline, Trigonometric B-Spline, method, Extended Cubic B-spline, Finite difference, finite element and finite volume method, B-Spline Functions and Partition method. The method needs much less computational work compared with traditional methods. It is shown that HPM is a very fast convergent, precise and cost efficient tool for solving boundary value problems. Therefore, this method can be seen as a promising and powerful tool for solving the second-order boundary value problems. Generally speaking, He's homotopy perturbation method is reliable and more efficient compared to other techniques.

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