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Research Article

Overlap Dimensions in Cyclic Tessellable Regular Polygons

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Abstract: In this study we study overlap dimensions in cyclic tessellable regular polygons. Overlap difference and area created by tessellable regular polygons inscribed in disks for covering play a significant role in computational geometry and signal interference in telecommunication network design. Regular triangles and squares are no exceptions except for optimality. We propose general formulae for computing the dimensions of a regular polygon inscribed in a disk. The study also leads to formulae for computing the overlap difference of tessellable regular polygons in disk covering. We realize that the cyclic regular hexagon has both optimal covering area and minimal overlap difference of 17.3 and 86.6% reduction over the original 100% disks size, respectively.

Keywords: Equilateral triangle and disks, hexagons overlap difference, squares

INTRODUCTION

Any tessellable regular polygon inscribed in a disk is a cyclic tessellable regular polygon. Equilateral triangles, squares and hexagons are known to be the only tillable regular polygons (Ding, 2010). Any regular polygon that can tile has the property of covering. These tessellable regular polygons have a lot of geometric and algebraic properties when inscribed in a disk with fixed radius. The algebraic relationship may be formulae for their overlap difference (d), area (A) or the relationship between the radius of the disk and the edges of the regular polygon. The geometric relationship discusses the number of dimensions, the number of segments when inscribed in a disk, line and rotational symmetry among others.

LITERATURE REVIEW

Covering with regular polygons has been one of the most fundamental and yet challenging issues in wireless network and found many applications such as routing and broad casting (Xu and Whang, 2011). Smith (1994) defines covering of the plane with copies of K as a family $\{K_i\}$ of sets congruent to K whose union is the plane. A family $\{K_i\}$ which is both packing and covering is called tiling.

Donkoh *et al*. (2014) emphasized with geometric proof that monohedral tilling's by regular polygons have varied overlap difference formulae. But the author failed to investigate the relationship between their dimensions when inscribed in a disk with fixed radius.

Lubachevsky and Graham (1997) computed the density (covering fraction) when 91 equal-size disks optimally packed in a hexagon to be closest to that of a circle with the same number of disks packed. This geometrically confirms the proof that hexagon approximates circle closely than any tessellable regular polygon

Paredes *et al*. (1998) stated that tiling with squares and triangles are very useful tools to study several structural and thermo dynamical properties of a wide variety of solids. The relationship between the overlaps created by equilateral triangles, squares and hexagons inscribed in disks has not yet been studied. This research paper aim at this relationship as well as the dimensions of a regular tessellable polygon that can be inscribed in disks with a fixed radius R_1.

COMPUTATIONAL EXPERIENCE

Consider the three tessellable regular polygons namely equilateral triangle, squares and hexagons inscribed in a disk of radius R_1 and centre *O*. The theorem below hold.

Theorem 1: In any circular disks of radius $R_{1,}$ hexagonal apothem r_1 and centre O, we can inscribe the following simultaneously:

a) A hexagon of side $R_{1,}$ or apothem $r_1 = \frac{R_1\sqrt{3}}{2}$
b) A square of side $s = \frac{2\sqrt{6}}{3} r_1$ or $R_1\sqrt{2}$
c) An equilateral triangle of sides $2r_1$ or $\sqrt{3}R_1$

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Proof: Figure 1 shows a disks with radius R_1 hexagonal apothem r_1, a square with dimensions s and an

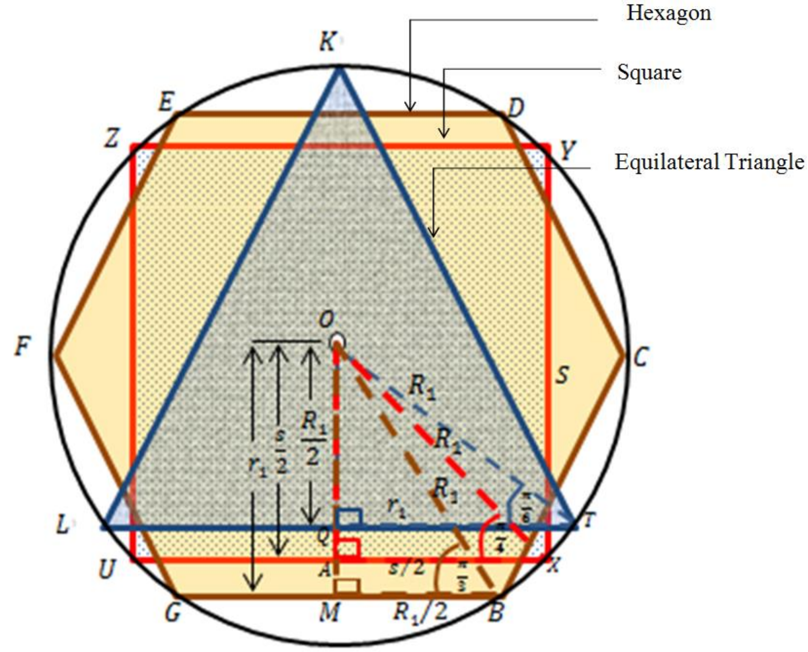


Fig. 1: Tessellable regular polygons inscribed in a disk

equilateral triangle with dimensions $\sqrt{3}R_1$. We find a relatiosn ship between the variables R_1, r_1 and s. Consider the following cases.

Case I: $s = f(R_1)$: We realize that triangle OQT is similar to triangle OMB, with $OT = BO = R_1$. From triangle OQT:

$$cos30^0 = \frac{QT}{R_1}$$

$$QT = \frac{\sqrt{3}}{2}R_1$$

$$\therefore LT = 2QT = \sqrt{3}R_1 \quad (1)$$

Considering triangle OAX:

$$R_1^2 = \frac{s^2}{4} + \frac{s^2}{4}$$
$$2R_1^2 = s^2$$

$$R_1 = \frac{s\sqrt{2}}{2} \quad (2)$$

$$\text{or } s = R_1\sqrt{2} \quad (3)$$

Case II: $r_1 = f(R_1)\ or\ f(s)$
Also, in ΔOMB which is similar to OQT:

$$R_1^2 = r_1^2 + \left(\frac{R_1}{2}\right)^2$$

$$3R_1^2 = 4r_1^2$$
$$R_1 = \frac{2r_1}{\sqrt{3}} = \frac{2r_1\sqrt{3}}{3} \text{ or} \quad (4)$$

$$r_1 = \frac{R_1\sqrt{3}}{2} \quad (5)$$

Substituting Eq. (4) into (3) we have:

$$S = \frac{2r_1\sqrt{3}}{3} \times \sqrt{2}$$

$$S = \frac{2\sqrt{6}}{3}r_1 \quad (6)$$

$$\text{Or } r_1 = \frac{3S}{2\sqrt{6}} = \frac{S\sqrt{6}}{4} \quad (7)$$

Then

$$AM = r_1 - \frac{S}{2} = \frac{S\sqrt{6}}{4} - \frac{S}{2}$$

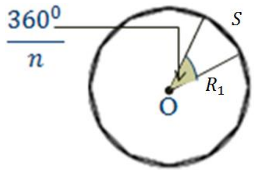
$$AM = \frac{1}{4}(\sqrt{6} - 2)s = f(s) \quad (8)$$

Also $AM = r_1 - \frac{S}{2}$

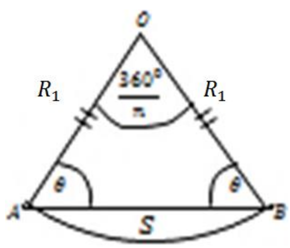
$$= r_1 - \frac{1}{2} \times \frac{2\sqrt{6}}{3}r_1$$

$$AM = \frac{1}{3}(3-\sqrt{6})r_1 = f(r_1) \quad (9)$$

Analogous to theorem 1 we generalize for all regular polygons as stated in theorem 2.



(a) $n-gon$ Inscribed in a disk



(b) Angle at the centre of an $n-gon$

Fig. 2: Polygon inscribed in a disk

Theorem 2: Given a circle of radius $R_{1,}$ we can inscribed a regular polygon of side length $2R_1 sin\left(\frac{180^0}{n}\right)$, where n is the number of sides of the regular polygon.

Proof: Suppose the regular polygon has n sides. Then the two successive radii connecting two internal angle is $\frac{360^0}{n}, n \geq 3$. Consider an n-gon inscribed in a disk as shown in Fig. 2:

Then $\theta = \frac{180^0 - \frac{360^0}{n}}{2}$

$$\theta = \left(90^0 - \frac{180^0}{n}\right)$$

Then, area of ΔAOB is equivalent to area of ΔOAB. mathematically:

$$\frac{1}{2}R_1^2 sin\left(\frac{360^0}{n}\right) = \frac{1}{2}SR_1 sin\left(90^0 - \frac{180^0}{n}\right)$$

$$S = \frac{\frac{1}{2}R_1^2 sin\left(\frac{360^0}{n}\right)}{\frac{1}{2}R_1 sin\left(90^0 - \frac{180^0}{n}\right)}$$

$$= \frac{R_1 sin\left(\frac{180^0}{n} + \frac{180^0}{n}\right)}{sin90^0 cos\left(\frac{180^0}{n}\right) - \cos 90^0 sin\left(\frac{180^0}{n}\right)}$$

$$= \frac{2R_1 sin\left(\frac{180^0}{n}\right) cos\left(\frac{180^0}{n}\right)}{sin90^0 cos\left(\frac{180^0}{n}\right)}$$

$$S = 2R_1 sin\left(\frac{180^0}{n}\right) \text{ For } n \geq 3 \quad (10)$$

Overlap difference in cyclic tessellable regular polygon: Tessellable regular polygons inscribed in disks overlap with difference (d). This overlap difference can be expressed in terms of R_1 or r_1 which can be compared to determine the best covering technique in GSM cell design or tilling in ancient or contemporary art. We consider the three tessellable regular polygons namely equi-triangular, square and hexagonal polygon.

Type I: Equi-triangular Polygon

Consider triangle OQT in Fig. 1 which is congruent to triangle OMB. Then $MB = OT = \frac{R_1}{2}$ and$OM = QT = r_1$. Thus, Eq. (4) and (5) holds:

$$R_1^2 = \left(\frac{R_1}{2}\right)^2 + (r_1)^2$$

$$\frac{3}{4}R_1^2 = r_1^2$$

$$R_1 = \frac{2}{\sqrt{3}}r_1 = \frac{2\sqrt{3}}{3}r_1$$

Case I: $d = f(R_1)$

$$d = 2\left(R_1 - \frac{R_1}{2}\right)$$

$$d = R_1 \quad (11)$$

Case II: $d = f(r_1)$

$$d = \frac{2r_1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}r_1 \quad (12)$$

Case III: $d = f(R_1, r_1)$:

$d = 2\left(R_1 - \frac{R_1}{2}\right)$ But $R_1 = \frac{2\sqrt{3}r_1}{3}$ as in Eq. (5):

$$= 2\left(R_1 - \frac{\frac{2\sqrt{3}r_1}{3}}{2}\right)$$

$$d = 2\left(R_1 - \frac{\sqrt{3}}{3}r_1\right)$$

$$d = \frac{2}{3}(3R_1 - \sqrt{3}r_1) \quad (13)$$

Type II: Square polygon: Consider square U, X, Y, Z in Fig. 1 with centre O and dimension s by s. We compute the overlap difference for this polygon and study the occupying difference ratio to that of a disk:

Case I: $d = f(R_1, r_1)$

$$d = 2(R_1 - OA) = 2AM$$
$$= 2\left(R_1 - \frac{S}{2}\right)$$
$$= 2\left(R_1 - \frac{1}{2} \times \frac{2\sqrt{6}}{3} r_1\right)$$
$$d = \tfrac{2}{3}\left(3R_1 - \sqrt{6}\, r_1\right) \tag{14}$$

Case II: $d = f(R_1)$:

$$d = \frac{2}{3}\left(3R_1 - \sqrt{6}\, r_1\right)$$
$$= \tfrac{2}{3}\left(3R_1 - \sqrt{6}\ \times R_1\sqrt{3}\right)$$
$$d = \left(2 - \sqrt{2}\right)R_1 \tag{15}$$

Case III: $d = f(s)$:

$$d = \left(2 - \sqrt{2}\right)R_1$$
$$= \left(2 - \sqrt{2}\right) \times \frac{S}{\sqrt{2}}$$
$$d = \left(\sqrt{2} - 1\right)S \tag{16}$$

Case IV: $d = f(r_1)$:

$$d = \frac{2}{3}\left(3R_1 - \sqrt{6}\, r_1\right)$$
$$= \frac{2}{3}\left(3 \times \frac{2r_1}{\sqrt{3}} - \sqrt{6}\, r_1\right)$$
$$d = \tfrac{2}{3}\left(2\sqrt{3} - \sqrt{6}\right)r_1 \tag{17}$$

Type III: Hexagon: Consider $BCDEFG$ in Fig. 1 with radiusR_1. It is observed that triangle OMB is similar to triangle OQT. Thus Eq. (4) and (5) holds. We consider the following cases:

Case I: $d = f(R_1, r_1) = 2(R_1 - r_1)$ (18)

Case II: $d = f(r_1)$:

$$d = 2\left(\frac{2\sqrt{3}r_1}{3} - r_1\right)$$
$$d = \tfrac{2}{3}\left(2\sqrt{3} - 3\right)r_1 \tag{19}$$

Case III: $d = f(R_1)$:

$$d = 2\left(R_1 - \frac{\sqrt{3}}{2}R_1\right)$$
$$d = \left(2 - \sqrt{3}\right)R_1 \tag{20}$$

Overlap area in cyclic tessellable regular polygon: Similarly, the areas of regular tessellable polygons inscribed in disks can be expressed as a function of the disk radius R_1, hexagonal apothem r_1 and overlap difference d. We consider the three regular tessellable polygons namely equi-triangular, square and hexagonal polygon:

Case I: Equi-triangular polygon: From ΔQLT in Fig. 1 we can calculate the area to be:

$$A_T = \tfrac{1}{2} \times \text{Base} \times \text{perpendicular height}$$
$$= \frac{1}{2} \times \sqrt{3}R_1 \times \left(R_1 + \frac{R_1}{2}\right)$$
$$= \tfrac{3\sqrt{3}}{4}R_1^2 \tag{21}$$

But in Eq. (4) $R_1 = \frac{2r_1\sqrt{3}}{3}$, implies:

$$A_T = \sqrt{3}\, r_1^2 \tag{22}$$

But from Eq. (12) $d_3 = \frac{2r_1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}r_1$, implies $r_1 = \frac{3d_3}{2\sqrt{3}}$:

$$A_T = \sqrt{3}\ \times \frac{9d_3^2}{6}$$
$$A_T = \tfrac{3\sqrt{3}}{4}d_3^2 \tag{23}$$

Case II: Square polygon: From square $UXYZ$ in Fig. 1, we can calculate the area to be:

$$A_s = s \times s$$
$$= R_1\sqrt{2} \times R_1\sqrt{2}$$
$$A_s = 2R_1^2 \tag{24}$$

Recall from Eq. (4) $R_1 = \frac{2r_1\sqrt{3}}{3}$, then:

$$A_s = \tfrac{8}{3}r_1^2 \tag{25}$$

Also, Eq. (14) indicates that $d_4 = \frac{2}{3}\left(2\sqrt{3} - \sqrt{6}\right)r_1$ which means $r_1 = \frac{\left(2\sqrt{3}+\sqrt{6}\right)}{4}d_4$.

Thus, our new area can be written in the form:

$$A_s = \frac{8}{3} \times \frac{\left(2\sqrt{3} + \sqrt{6}\right)^2}{16}d_4^2$$
$$A_s = \left(3 + 2\sqrt{2}\right)d_4^2 \tag{26}$$

Case III: Hexagonal polygon: Consider hexagon $BCDEFGU$ as shown in Fig. 1. We have the area to be:

$$A_H = 6 \times \frac{1}{2} \times R_1 \times R_1 \times sin60^0$$

$$A_H = \frac{3\sqrt{3}}{2} R_1^2 \quad (27)$$

Recall from Eq. (4) $R_1 = \frac{2r_1\sqrt{3}}{3}$, then:

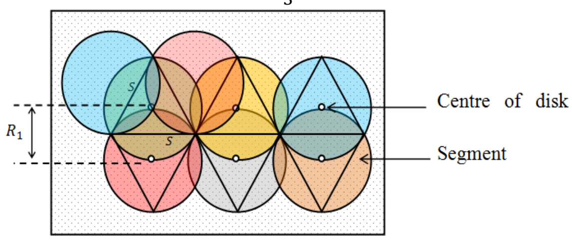


Fig. 3: Equi-triangular tilling in disks

$$A_H = \frac{3\sqrt{3}}{2} \times \frac{2r_1\sqrt{3}}{3}$$

$$A_H = 2\sqrt{3}\, r_1^2 \quad (28)$$

Also, in Eq. (19) $d_6 = \frac{2}{3}(2\sqrt{3} - 3)r_1$, implies $r_1 = \frac{(2\sqrt{3}+3)}{2} d_6$ where d_6 is the overlap difference of hexagon inscribed in a circle. Then:

$$A_H = 2\sqrt{3} \times \frac{\left(2\sqrt{3} + 3\right)^2}{4} d_6^2$$

$$A_H = \frac{3}{2}(12 + 7\sqrt{3}) d_6^2 \quad (29)$$

Theorem 2: Disks have an overlap difference of $2R_1$ or $\frac{4\sqrt{3}}{3} r_1$ for covering since it does not tile. The area and overlap difference are respectively $\pi R_1^2 = \frac{4}{3}\pi r_1^2$ and $\frac{\pi}{4} d_\infty^2$.

Proof: From Fig. 1, we know that equi-triangular polygon has an overlap difference of R_1 as in Eq. (11). But the diameter of the disks is $2R_1$ which is twice that of the overlap difference of an equi-triangular polygon. Figure 3 illustrates equi-triangular tile.

With side$s$, apothem $\frac{R_1}{2}$ inscribed in a disk with radius R_1.

Figure 3 six (6) equal segments completely cover a disks circumscribed on equi-triangular tilling whereas 3 equal segments completely covers each equi-triangular tile. So the relationship between their overlap difference in covering will be $2R_1$ is to R_1 respectively. Thus disks cover with overlap difference of $2R_1$.

It follows from Eq. (4) that $R_1 = \frac{2r_1\sqrt{3}}{3}$. Hence diameter of circle (overlap difference of regular polygon of n sides as n approaches infinity), d_∞ is:

$$d_\infty = 2 \times \frac{2r_1\sqrt{3}}{3}$$

$$d_\infty = \frac{4\sqrt{3}}{3} r_1 \quad (30)$$

Similarly, the area of circle is:

$$A_c = \pi R_1^2$$

$$A_c = \frac{4\pi}{3} r_1^2 \quad (31)$$

From Eq. (30) $r_1 = \frac{3d_\infty}{4\sqrt{3}}$ then:

$$A_c = \frac{4\pi}{3} \times \frac{9d_\infty^2}{48}$$

$$A_c = \frac{\pi}{4} d_\infty^2 \quad (32)$$

Ratio of overlap difference and area for tesselable regular polygons inscribed in disks: Table 1 shows the relationship between the overlap difference and area for three tessellable regular polygons inscribed in a disk with radius R_1, hexagonal apothem r_1 and their occupying ratio or covering fraction to that of the disks.

From Table 1 as radius increases overlapped area decreases according to inverse square law because curvature of circular shape of signal gets larger and larger. We deduce that for a hexagon (H), square (S) and equi-triangular (T) polygon, the following inequality holds for their overlap difference terms of:

a) Radius of disks(R_1): $H_{(2-\sqrt{3})R_1} < S_{(2-\sqrt{2})R_1} < T_{R_1}$
b) Radius of disks and apothem (R_1, r): $H_{2(R_1 - r_1)} < S_{2/3(3R_1 - \sqrt{6}\, r_1)} < T_{2/3(3R_1 - \sqrt{3} r_1)}$
c) Apothem (r): $H_{\frac{2}{3}(2\sqrt{3}-3)r_1} < S_{\frac{2}{3}(2\sqrt{3}-\sqrt{6})r_1} < T_{\frac{2\sqrt{3}}{3}r_1}$.

Thus, (a), (b) and (c) implies that the hexagon has the least overlap width and therefore is best suited for geometric covering using polygons. Hexagonal tiling with least overlap difference implies least overlap area or wide non overlapping area. It is expected that the hexagonal covering define in terms of the overlap area will be greater than that of a square and an equi-triangular polygon. This coverage area defined in terms of:

d) Radius of disks (R_1): $H_{\frac{3\sqrt{3}}{2}R_1^2} > S_{2R_1^2} > T_{\frac{3\sqrt{3}}{4}}$.

e) Overlap difference (d): $H_{\frac{3}{2}(12+7\sqrt{3})d^2} > S_{(3+2\sqrt{2})d^2} > T_{\frac{3\sqrt{3}}{4}d^2}$

It is evident that regular hexagon has the maximum coverage area 82.7% of disk area or least overlap difference 13% of disk difference and is therefore the best geometric object for optimal disk covering in a plane.

Table 1: Each overlap difference, area and their ratio for cyclic tessellable regular polygons

Tessellable regular Polygon Vrs. Disks	Disks (D)	Triangle (T)	Square (S)	Hexagon (H)
$d = f(R_1, r_1)$	non	$\frac{2}{3}(3R_1 - \sqrt{3}r_1)$	$\frac{2}{3}(3R_1 - \sqrt{6}\,r_1)$	$2(R_1 - r_1)$
$d = f(R_1) = g(r_1)$	$2R_1$	R_1	$(2-\sqrt{2})R_1$	$(2-\sqrt{3})R_1$
	$\frac{4\sqrt{3}}{3}r_1$	$\frac{2\sqrt{3}}{3}r_1$	$\frac{2}{3}(2\sqrt{3}-\sqrt{6})r_1$	$\frac{2}{3}(2\sqrt{3}-3)r_1$
Overlap difference ratio $d_* = \left(\frac{D}{D}:\frac{T}{D}:\frac{S}{D}:\frac{H}{D}\right)$	100%	50%	29.3%	13.4%
Area $= f(R_1) = g(r_1)$	πR_1^2	$\frac{3\sqrt{3}}{4}R_1^2$	$2R_1^2$	$\frac{3\sqrt{3}}{2}R_1^2$
	$\frac{4}{3}\pi r_1^2$	$\sqrt{3}r_1^2$	$\frac{8}{3}r_1^2$	$\sqrt{3}r_1^2$
Overlap area ratio $d_0 = \left(\frac{D}{D}:\frac{T}{D}:\frac{S}{D}:\frac{H}{D}\right)$	100%	41.3%	63.7%	82.7%
$Area = f(d)$	$\frac{\pi}{4}d_\infty^1$	$\frac{3\sqrt{3}}{4}d_3^2$	$(3+2\sqrt{2})d_4^2$	$\frac{3}{2}(12+7\sqrt{3})d_6^2$

DISCUSSION OF RESULTS

Three tessellable regular polygons with hexagonal apothem r_1 were inscribed in a disk of radius R_1 resulted in a hexagon of dimension R_1 $\left(\text{or apothem } \frac{R_1\sqrt{3}}{2}\right)$, a square with side $\frac{2\sqrt{6}}{3}r_1$ $\left(\text{or } R_1\sqrt{2}\right)$ or an equi-triangular polygon with side $2r_1$ $\left(\text{or } \sqrt{3}R_1\right)$. This was generalize to regular polygons with dimension $2R_1 sin\left(\frac{180^0}{n}\right)$ for $n \geq 3$. We computed the overlap difference for these regular polygons. As a result hexagon overlap with a difference of $\frac{2}{3}(2\sqrt{3}-3)$ r_1 (or (2-√3)R_1), square overlap with $\frac{2}{3}(2\sqrt{3}-\sqrt{6})r_1$ (or (2-√2)R_1) and an equilateral triangle overlap with difference of $\frac{2\sqrt{3}}{3}r_1$ $(or\ R_1)$. We realize that hexagon occupies the least overlap difference of 13.4% as compared to 29.3% for a square and 50% for a regular triangle. Overlap area calculated to be $\frac{3\sqrt{3}}{2}R_1^2$ for hexagonal tessellation is 82.7% approximate to the area of a disks. That of square and equi-triangular polygon approximate at 63.7 and 41.3% respectively. These values could be used to judge the honeycomb conjecture that hexagon is the most efficient way to tessellate the plane in terms of the total perimeter per area coverage. Overlap area expressed as a function of the difference could not be uniquely formulated for all regular polygons since there is no rule connecting the number of sides and the overlap difference.

CONCLSUION

Calculation of amount of overlapping coverage area is important in cellular system as the total amount of signal interference depends on the overlapping coverage area. This amount of signal interference plays an important role in making the decision of handover. A closed form of an optimal tessellable hexagonal coverage with overlap difference of $(2-\sqrt{3})R_1$ is presented in this study and the calculation of minimum overlapping coverage area $\frac{3\sqrt{3}}{2}R_1^2$ as compared to that of a square or equi-triangular tile in Table 1. We use geometry of regular tessellable polygons inscribed in disks to obtain the optimality and it is the first study that combines the three tessellable regular polygons inscribed in a single disk to arrive at the least overlap difference or optimal coverage area. The study also shows that in a disk of radius R_1 and hexagonal apothem r_1 we can inscribe a regular polygon of dimension $2R_1 sin\left(\frac{180^0}{n}\right)$. This formula helps geometrically in least time complexity for inscribing a regular polygon in a disk. Specifically, a hexagon of side $R_1\ or$ $\left(\text{or apothem } r_1 = \frac{R_1\sqrt{3}}{2}\right)$ a square of side $\frac{2\sqrt{6}}{3}r_1$ $\left(\text{or } R_1\sqrt{2}\right)$ or an equilateral triangle of sides $2r_1$ (or $\sqrt{3}R_1$) could be obtain.

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