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Research Article

Transitive 5-Groups of Degree $5^2 = 25$

E. Apine, B.N. Jelten and E.N. Homti,
Department of Mathematics, University of Jos, PMB 2084, Jos, Nigeria

Abstract: In this study we achieve a classification of transitive 5-groups of degree 25and we realize and identify some of the unique properties that are associated with them.

Keywords: Classification, degree, isomorphism, p-groups, transitive

INTRODUCTION

Let G be a group acting on a non-empty set Ω and the letter p represents an arbitrary but fixed prime number and in our case $p = 5$. The action of G on Ω is said to be transitive if for any α, β in Ω there exists some g in G such that $\beta = \alpha g$ In this case $|\Omega|$ is called the degree of G on Ω. Audu (1988 a to c), Audu (1989 a, b) determined the number of transitive p-groups of degree p^2 and, Apine (2002), Apine and Jelten (2014) achieved a classification of transitive and faithful p-groups (Abelian and Non-abelian) of degrees at most p^3 whose centre is elementary Abelian of rank two. In this study, we determine, up to equivalence, the actual transitive p-groups (Abelian and Non-abelian) of degree p^2 for $p = 5$ and achieve a classification of transitive 5 groups of degree $5^2 = 25$ (Audu *et al*., 2006, Audu and Apine, 1993 and Audu, 1991a and b).

RESULTS

Transitive 5-groups of degree $5^2 = 25$: In the procedure outlined below we rely heavily on the algebraic computer software GAP (Groups, Algorithms and Pragramming) to obtain both the presentations and the generators of the groups under investigation.

Let G be a transitive 5-group of degree 5^2. Then $G \leq Sym(\Omega)$, where $\Omega = \{1, 2, \ldots, 25\}$ and as $|Sym(25)| = 25! = 2^{22}.3^{10}.5^6.7^3.11^2.13.17.19.23$, it follows that:

$|G| = 5^n$, n =1, 2, ..., 6.

Clearly $n \neq 1$ and when n = 2, then $|G| = 25$ and for transitivity:

$|\alpha^G| = 25$, $|G_\alpha| = 1$, $\forall\ \alpha \in \Omega$

In case G is abelian and either $G \cong C_{25}$ or $G \cong C_5 \times C_5$.

If $G \cong C_{25}$, then $G = G_{1,2} = \langle a \rangle$, with generator, say, a = (1, 2, 3, ..., 25)

If $G \cong C_5 \times C_5$, then $G = G_{2,2} = \langle a, b: a^5 = 1, b^5 = 1, ab = ba \rangle$ with generators.

a = (1, 2, ..., 5) (6, 7, ..., 10) (11, 12, ..., 15) (16, 17, ..., 20) (21, 22, ..., 25) and

b = (1, 10, 14, 25, 17) (2, 6, 15, 21, 18) (3, 7, 11, 22, 19) (4, 8, 12, 23, 20) (5, 9, 13, 24, 16)

Clearly $G_{1,2}$ and $G_{2,2}$ are transitive on Ω and we have

Lemma 1: There are, up to isomorphism, 2 transitive 5-groups of degree 25 and order 25, namely the abelian groups $G_{1,2}$ and $G_{2,2}$ described above.

When n = 3, then $|G| = 125$ and for transitivity, $|\alpha^G| = 25$, $|G_\alpha| = 5$, $\forall\ \alpha \in \Omega$.

Thus G is non-abelian and we have the following possibilities for G:

$$G \cong G_{1,3} = \langle a, b: a^{25} = 1, b^5 = 1, ab = ba^6 \rangle \text{ or } G \cong G_{2,3} = \langle G_{2,2}, c \rangle$$

where $c^5 = 1$, $G_{2,2} \trianglelefteq G_{2,3}$.

For $G_{1,3}$, we take as generators a = (1, 2, ..., 25) and b = (1, 6, 11, 16, 21) (2, 12, 22, 7, 17) (3, 18, 8, 23, 13) (4, 24, 19, 14, 9).

For $G_{2,3}$, we have as presentation:

$G_{2,3} = \langle a, b, c: a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb \rangle$ with generators a, b the same as those of $G_{2,2}$ and c = (1, 18, 7, 8, 16) (2, 11, 12, 5, 10) (3, 4, 24, 17, 21) (6, 22, 23, 9, 14) (13, 25, 15, 19, 20).

Hence we have:

Lemma 2: There are, up to isomorphism, two transitive 5-groups of degree 25 and order 125, namely the non-abelian groups $G_{1,3}$ and $G_{2,3}$ described above.

When n = 4, then $|G| = 625$ and for transitivity, $|\alpha^G| = 25$, $|G_\alpha| = 25$, $\forall\ \alpha \in \Omega$.

Corresponding Author: E. Apine, Department of Mathematics, University of Jos, PMB 2084, Jos, Nigeria

Table 1: The Number of Transitive 5-Groups of Degree $5^2 = 25$, up to Isomorphism

N	$\lvert G \rvert = 5^n$	Number of transitive abelian 5-groups of degree 25, up to isomorphism	Number of transitive non-abelian 5-groupsof degree 25, up to isomorphism	Number of transitive 5-groups of degree 25, up to isomorphism
1	5	0	0	0
2	25	2	0	2
3	125	0	2	2
4	625	0	2	2
5	3125	0	2	2
6	15625	0	1	1
Total		2	7	9

Thus G is non-abelian and we have the following possibilities for G:

$G \cong G_{1,4} = \langle G_{1,3}, c\rangle$ with $c^5 = 1$, $G_{1,3} \trianglelefteq G_{1,4}$ or $G \cong G_{2,4} = \langle G_{2,3}, d\rangle$ with $d^5 = 1$, $G_{2,3} \trianglelefteq G_{2,4}$

For the case $G_{1,4}$, we have a presentation:

$G_{1,4} = \langle a, b, c: a^{25} = 1, b^5 = 1, ab = ba^6, c^5 = 1, ac = cab, bc = cb\rangle$ with generators a, b the same as those of $G_{1,3}$ and c = (1, 6, 11, 16, 21) (2, 17, 7, 22, 12) (3, 8, 13, 18, 23) (see *Gap*-programme 3).

For $G_{2,4}$, we have as presentation:

$G_{2,4} = \langle a, b, c, d: a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb, d^5 = 1, ad = dbc, bd = db, cd = da^4b^3c^2\rangle$ with generators a, b, c the same as those of $G_{2,3}$ and d = (1, 3, 5, 15, 23) (2, 8, 25, 22, 24) (4, 14, 11, 13, 18) (6, 12, 17, 19, 16) (7, 9, 21, 20, 10).

Hence:

Lemma 3: There are, up to isomorphism, 2 transitive 5-groups of degree 25 and order 625, namely the non-abelian groups $G_{1,4}$ (of exponent 25) and $G_{2,4}$ (of exponent 5) described above.

When n = 5, then $\lvert G\rvert = 3125$ and for transitivity, $\lvert \alpha^G\rvert = 25$, $\lvert G_\alpha\rvert = 125$, $\forall\ \alpha \in \Omega$.

Thus G is non-abelian and we have the following possibilities for G:

$G \cong G_{1,5} = \langle G_{1,4}, d\rangle$ with $d^5 = 1$, $G_{1,4} \trianglelefteq G_{1,5}$ or $G \cong G_{2,5} = \langle G_{2,4}, e\rangle$ with $e^5 = 1$, $G_{2,4} \trianglelefteq G_{2,5}$

For the case $G_{1,5}$, we have a presentation:

$G_{1,5} = \langle a, b, c, d: a^{25} = 1, b^5 = 1, ab = ba^6, c^5 = 1, ac = cab, bc = cb, d^5 = 1, ad = dab^3c, bd = db, cd = dc\rangle$ with generators a, b, c the same as those of $G_{1,4}$ and d = (1, 6, 11, 16, 21) (4, 14, 24, 9, 19) (5, 15, 25, 10, 20) (obtained from PROGRAMME 3).

For $G_{2,5}$, we have as presentation:

$G_{2,5} = \langle a, b, c, d, e: a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb, d^5 = 1, ad = dbc, bd = db, cd = da^4b^3c^2, e^5 = 1, ae = ea^2bcd^4, be = eb, ce = eab^3c^2d^4, de = ea^3bcd^4\rangle$ with generators a, b, c, d the same as those of $G_{2,4}$ and e = (1, 25, 10, 17, 14) (2, 21, 6, 18, 15) (3, 11, 19, 7, 22) (4, 12, 20, 8, 23). Thus:

Lemma 4: There are, up to isomorphism, 2 transitive 5-groups of degree 25 and order 3125, namely the non-abelian groups $G_{1,5}$ (of exponent 25) and $G_{2,5}$ (of exponent 5) described above.

When n = 6, then $\lvert G\rvert = 15625$ and for transitivity, $\lvert \alpha^G\rvert = 25$, $\lvert G_\alpha\rvert = 625$, $\forall\ \alpha \in \Omega$.

Thus G is non-abelian and we have the following possibilities for G:

$G \cong G_{1,6} = \langle G_{1,5}, e\rangle$ with $e^5 = 1$, $G_{1,5} \trianglelefteq G_{1,6}$ or $G \cong G_{2,6} = \langle G_{2,5}, f\rangle$ with $f^5 = 1$, $G_{2,5} \trianglelefteq G_{2,6}$

For the case $G_{1,6}$, we have a presentation:

$G_{1,6} = \langle a, b, c, d, e: a^{25} = 1, b^5 = 1, ab = ba^6, c^5 = 1, ac = cab, bc = cb, d^5 = 1, ad = dab^3c, bd = db, cd = dc, e^5 = 1, ae = eab^3cd^3, be = eb; ce = ec, de = ed\rangle$ with generators a, b, c, d the same as those of $G_{1,5}$ and e = (1, 6, 11, 16, 21) (4, 19, 9, 24, 14) (5, 25, 20, 15, 10).

For $G_{2,6}$, we have as presentation:

$G_{2,6} = \langle a, b, c, d, e: a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb, d^5 = 1, ad = dbc, bd = db, cd = da^4b^3c^2, e^5 = 1, ae = ea^2bcd^4, be = eb, ce = eab^3c^2d^4, de = ea^3bcd^4, f^5 = 1, af = fa^4b^4c^2e, bf = f b, cf = fa^3c^3e, df = fabc^2d^2e^2, ef = fe\rangle$ with generators a, b, c, d, e the same as those of $G_{2,5}$ and f = (1, 10, 14, 25, 17) (2, 21, 6, 18, 15) (4, 23, 8, 20, 12).

We notice here that $G_{2,6}$ is of exponent 25 and that $G_{1,6} \cong G_{2,6}$. Consequently:

Lemma 5: There is, up to isomorphism, only one transitive 5-group of degree 25 and order 15625, namely the non-abelian group $G_{1,6}$(of exponent 25) described above.

We summarize our findings as in Table 1 and we state:

Proposition: There are, up to isomorphism, 9 transitive 5-groups of degree 5^2, 2 of these are abelian and of the remaining 7 non-abelian, 4 are of exponent 25 and 3 are of exponent 5.

Programme 3:

```
Gap>s25: = Symmetric Group (25)
Gap>a: = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25)
Gap>b: = (1, 6, 11, 16, 21) (2, 12, 22, 7, 17) (3, 18, 8, 23, 13) (4, 24, 19, 14, 9)
Gap>H: = Subgroup (s25, [a, b])
Gap. Centa: = Centralizer (s25, a) ;; centb: = Centralizer (s25,b)
Gap>x: = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25)
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Gap>y: = (1, 2, 3, 4, 5) (6, 7, 8, 9, 10)
Gap>K: = Subgroup (s25, [x, y])
Gap>int: = Intersection (K, centb)
Gap>diff: = Difference (int, H)
Gap> req: = []
Gap>for c in diff do
>if Order (int, c) = 5 then
>if Order (int, c)<>25 then
>if Size (Subgroup (s25, [a, b, c])) = 625 then
>Add (req, c)
>fi
>fi
>fi
>od
gap>req
gap>Size (req)
gap>100
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REFERENCES

Apine, E. and B.N. Jelten, 2014. Trends in transitive p-groups and their defining relations. J. Mathe. Theor. Model., 4(11): 192-209.

Audu, M.S., 1988a. The structure of the permutation modules for transitive p-groups of degree p^2. J. Algebra, 117: 227-239.

Audu, M.S., 1988b. The structure of the permutation modules for transitive Abelian groups of prime-power order. Nigerian J. Mathe. Appli., 1: 1-8.

Audu, M.S., 1988c. The number of transitive p-groups of degree p^2. Adv. Modell. Simulat. Enterprises Rev., 7(4): 9-13.

Audu, M.S., 1989a. Groups of prime-power order acting on modules over a modular field. Adv. Modell. Simulat. Enterprises Rev., 9(4): 1-10.

Audu, M.S., 1989b. Theorems about p-groups. Adv. Modell. Simulat. Enterprises Rev., 9(4):11-24.

Audu, M.S., 1991a. The loewy series associated with transitive p-groups of degree p^2. Abacus, 2(2): 1-9.

Audu, M.S., 1991b. On transitive permutation groups. Afrika Mathmatika J. African Mathe. Union, 4(2): 155-160.

Audu, M.S. and S.U. Momoh, 1993. An upper bound for the minimum size of generating set for a permutation group. Nigerian J. Mathe. Appl., 6: 9-20.

Audu, M.S., A. Afolabi and E. Apine, 2006. Transitive 3-groups of degree 3^n (n =2, 3).Kragujevac J. Mathe., 29: 71-89.

Apine, E., 2002. On Transitive p-Groups of Degree at most p^3. Ph.D. Thesis, University of Jos, Jos.