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Research Article

An Extended Generalized Power Lindley Distribution

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Abstract: In this study, a five-parameter distribution so-called the kumaraswamy Generalized power Lindley is defined and studied. The new distribution contains, as special sub models, several important distributions, such as the kumaraswamy generalized Lindley, kumaraswamy Lindley, generalized Lindley. We derive the moments moment generating function, conditional moment and mean residual lifetime are derived. We propose the method of maximum likelihood for estimating the model parameters. Finally, real data examples are discussed to illustrate the usefulness and applicability of the proposed distribution.

Keywords: Generalized power lindley, kumaraswamy distribution, maximum likelihood estimation, momemts

INTRODUCTION

The Lindley distribution was introduced by Lindley (1958) as a new distribution useful to analyze lifetime data especially in applications modeling stress-strength reliability. Ghitany *et al*. (2008) studied the properties of the Lindley distribution under a carefully mathematical treatment. They also showed in a numerical example that the Lindley distribution gives better modeling for waiting times and survival times data than the exponential distribution. The use of the Lindley distribution could be a good alternative to analyze lifetime data within the competing risks approach as compared with the use of standard Exponential or even the Weibull distribution commonly used in this area.

Lindley (1958) introduced a one- parameter distribution, known as Lindley distribution, given by its probability density function:

$$f(x,\theta) = g(x,\theta) = \left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{\{-\theta x\}};\ x > 0, \theta > 0, \tag{1}$$

The cumulative distribution function (cdf) of Lindley distribution is obtained as:

$$F(x,\theta) = 1 - e^{\{-\theta x\}\left[1+\left(\frac{\theta x}{\theta+1}\right)\right]}, x > 0, \theta > 0 \tag{2}$$

In the context of reliability studies, Ghitany *et al*. (2013) proposed the power Lindley distribution which is extension of Lindley distribution which offers a more flexible distribution for modeling lifetime data, namely in reliability, in terms of its failure rate shapes. It can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates. The cumulative distribution function (cdf) of power Lindley distribution is given by:

$$F(x,\alpha,\theta) = 1 - \left(1 + \left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}} \tag{3}$$

And the probability density function as follows:

$$f(x,\alpha,\theta) = \left(\frac{\alpha^2\theta}{1+\alpha}\right)\left(1 + x^{\{\theta\}}\right)x^{\{\theta-1\}e^{\left\{-\alpha x^{\{\theta\}}\right\}}} \tag{4}$$

Pararai *et al*. (2015) introduced a generalization of Power Lindley called generalized (or exponentiated) power Lindley (GPL). This distribution represents a more flexible model for the lifetime data.

A random variable X is said to have the generalized power Lindley (GPL) distribution with three parameters α, θ and β, if it has the cumulative distribution function:

$$G(x,\alpha,\theta,\beta) = \left[1 - \left(1 + \left(\frac{\alpha x^{\theta}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\theta}\}}\right]^{\beta}, x > 0 \tag{5}$$

The corresponding probability density function (pdf) is given by:

$$g(x,\alpha,\theta,\beta) = \left(\frac{\theta\alpha^2\beta}{1+\alpha}\right)\left(1 + x^{\{\theta\}}\right)x^{\{\theta-1\}}e^{\{-\alpha x^{\{\theta\}}\}} \times \left[1 - \left(1 + \left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{\beta-1\}}, x > 0, \alpha, \theta, \beta > 0. \tag{6}$$

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The distribution introduced by Kumaraswamy (1980), also referred to as the "minimax" distribution, is not very common among statisticians and has been little explored in the literature, nor its relative interchangeability with the beta distribution has been widely appreciated. We use the term K_w distribution to denote the Kumaraswamy distribution. The Kumaraswamy K_w distribution is not very common among statisticians and has been little explored in the literature. Its cumulative distribution function (cdf) is given by:

$$F_{\{X|(a,b)\}}(x) = 1 - \left(1 - x^{\{a\}}\right)^{\{b\}}, 0 < x < 1 \qquad (7)$$

where, $a > 0$ and $b > 0$ are shape parameters. Equation (7) compares extremely favorably in terms of simplicity with the beta cdf which is given by the incomplete beta function ratio. The corresponding probability density function (pdf) is:

$$f_{\{X|(a,b)\}}(x) = abx^{\{a-1\}}\left(1 - x^{\{a\}}\right)^{\{b-1\}} \qquad (8)$$

The K_w pdf has the same basic shape properties of the beta distribution: $a > 1$ and $b > 1$ (unimodal); $a < 1$ and $b < 1$(uniantimodel); $a > 1$ and $b \leq 1$ (increasing); $a \leq 1$ and $b > 1$(decreasing); a = 1 and b = 1 (constant). It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones (2009) explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and K_w distributions. However, the beta distribution has the following advantages over theK_w distribution: simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, we combine the works of Kumaraswamy (1980) and Cordeiro and de Castro (2011) to derive some mathematical properties of a new model, called the K_w −G distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by:

$$F_{\{X|(a,b)\}}(x) = 1 - \left[1 - G(x)^{\{a\}}\right]^{\{b\}} \qquad (9)$$

where $a > 0$ and $b > 0$ are two additional shape parameters which aim to govern skewness and tail weight of the generated distribution. An attractive feature of this distribution is that the two parameters a and b can afford greater control over the weights in both tails and in its Centre. The $K_w - G$ distribution can be used quite effectively even if the data are censored. The corresponding probability density function (pdf) is:

$$f_{\{X|(a,b)\}}(x) = abg(x)G(x)^{\{a-1\}\left[1-G(x)^{\{a\}}\right]^{\{b-1\}}} \qquad (10)$$

The density family (10) has many of the same properties of the class of beta-G distributions (Eugene *et al*., 2002), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special K_w-G distributions can be generated as follows: K_w −Weibull (Cordeiro *et al*., 2010), General results for the Kumaraswamy-G distribution (Nadarajah *et al*., 2012). K_w- generalized gamma (de Pascoa *et al*., 2011), K_w- Birnbaum-Saunders (Saulo *et al*., 2012) and K_w-Gumbel (Cordeiro *et al*., 2012) distributions are obtained by taking G(x) to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions. K_w exponentiated Pareto distribution (Elbatal, 2013). K_w-Quasi Lindley Distribution (Elbatal and Elgarhy, 2014). Recently, K_w- modified inverse Weibull distribution and its application, among several others. Hence, each new $K_w - G$distribution can be generated from a specified G distribution.

This study seeks to providea new five-parameter distribution so-called the kumaraswamy Generalized power Lindley is defined and studied. The new distribution contains, as special sub models, several important distributions, such as the kumaraswamy generalized Lindley, kumaraswamy Lindley, generalized Lindley. We derive the moments moment generating function, conditional moment and mean residual lifetime are derived. We propose the method of maximum likelihood for estimating the model parameters.

KUMARASWAMY GENERALIZED POWER LINDLEY DISTRIBUTION

In this section, we introduce the five-parameter Kumaraswamy generalized Power Lindley (KGPL) distribution. Using (5) in (9), the cdf of the (KGPL) distribution can be written as

$$(x, \alpha, \theta, \beta, a, b) = 1\text{-} \left\{1 - \left[1 - \left(1 + \left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right) e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{a\beta\}}\right\}^{\{b\}} \qquad (11)$$

The corresponding probability density function given by:

$$f_{KGPL}(x,\alpha,\theta,\beta,a,b) = abg(x)G(x)^{\{a-1\}}\left[1-G(x)^{\{a\}}\right]^{\{b-1\}}$$
$$= \left(\frac{ab\theta\alpha^2\beta}{1+\alpha}\right)\left(1+x^{\{\theta\}}\right)x^{\{\theta-1\}}e^{\{-\alpha x^{\{\theta\}}\}} \times$$
$$\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{\beta a-1\}}$$
$$\times\left\{1-\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{a\beta\}}\right\}^{\{b-1\}} \quad (12)$$

Here and henceforth, let $X \sim KGPL(\alpha,\theta,\beta,a,b)$ be a random variable with density function (12). The failure (hazard) rate function is given by:

$$h(x,\alpha,\theta,\beta,a,b) = \left(\frac{f_{KGPL}(x,\alpha,\theta,\beta,a,b)}{F(x,\alpha,\theta,\beta,a,b)}\right) = \left(\frac{ab\theta\alpha^2\beta}{1+\alpha}\right)\left(1+x^{\{\theta\}}\right)x^{\{\theta-1\}}e^{\{-\alpha x^{\{\theta\}}\}} \times$$
$$\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{\beta a-1\}} \times$$
$$\left(\frac{1}{\left\{1-\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{a\beta\}}\right\}}\right) \quad (13)$$

Also, using (11) and (12) we get the reversed failure (or reversed hazard) rate function which is given by $\tau(x) = \left(\frac{f(x)}{F(x)}\right)$ as:

$$\tau(x) = \left(\frac{ab\theta\alpha^2\beta}{1+\alpha}\right)\left(1+x^{\{\theta\}}\right)x^{\{\theta-1\}}e^{\{-\alpha x^{\{\theta\}}\}} \times$$
$$\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{\beta a-1\}} \times$$
$$\frac{\left\{1-\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{a\beta\}}\right\}^{\{b-1\}}}{1-\left\{1-\left[1-\left(1+\left(\frac{\alpha x^{\{\theta\}}}{\alpha+1}\right)\right)e^{\{-\alpha x^{\{\theta\}}\}}\right]^{\{a\beta\}}\right\}^{\{b\}}} \quad (14)$$

Figure 1 depicts the plots of the probability density and hazard function of KGPL distribution for some different values of the parameters.

Special cases of the KGPL distribution: The kumasraswamy generalized power Lindley is very flexible model that approaches to different distributions when its parameters are changed. The KGPL distribution contains as special-models the following well known distributions. If X is a random variable with pdf (12), we use the notation $X \sim KGPL(\alpha,\theta,\beta,a,b)$ then we have the following cases.

- If $= b = 1$, then (12) reduces to the generalized power Lindley which introduced by Pararai *et al*. (2015).
- If $\beta = 1$, we get kumaraswamy power Lindley distrribution.
- For $\theta = 1$ we get the kumaraswamy generalized Lindley distrribution which introduced by Oluyede *et al*. (2015).
- kumaraswamy Lindley distribution arises as a special case of KGPL by taking $\theta = \beta = 1$.
- Applying $a = b = \theta = 1$ we can obtain the generalized Lindley distrribution which introduced by Nadarajah *et al*. (2011).
- If $a = b = \theta = \beta = 1$we get Lindley distribution which introduced by Lindley (1958).

Expansion for the density function: In this subsection, we present some representations of pdf of Kumaraswamy generalized power Lindley distribution. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and |z|<1, then:

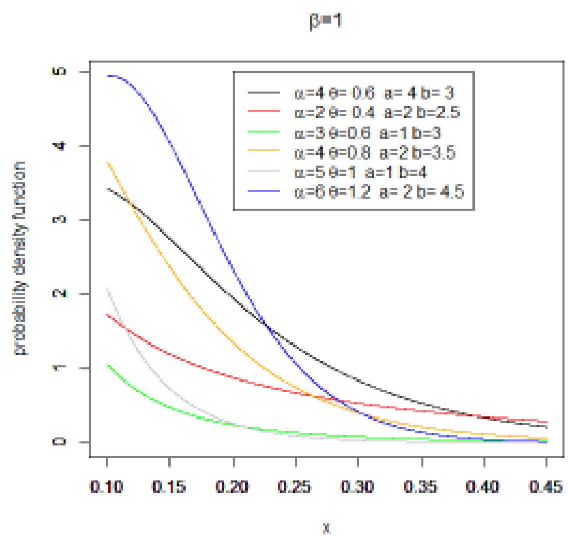

Fig. 1a: The pdf function of KGPL for some parameter values

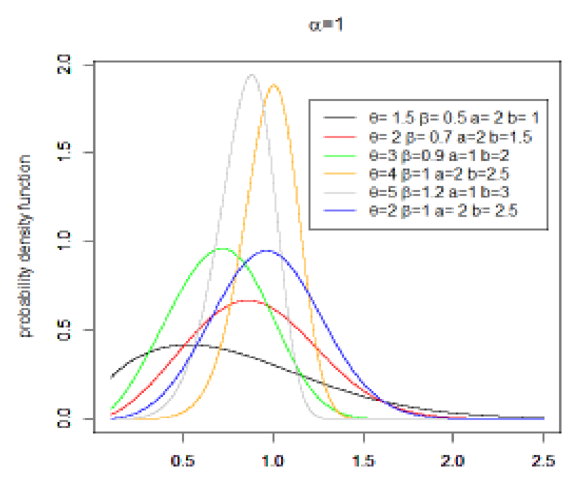

Fig. 1b: The pdf function of KGPL for some parameter values

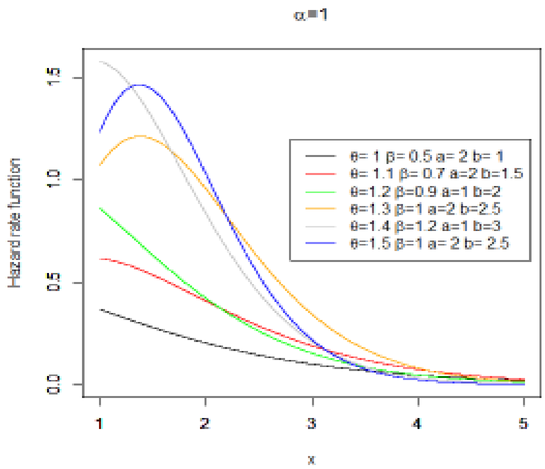

Fig. 1c: The hazard function of KGPL for some parameter values

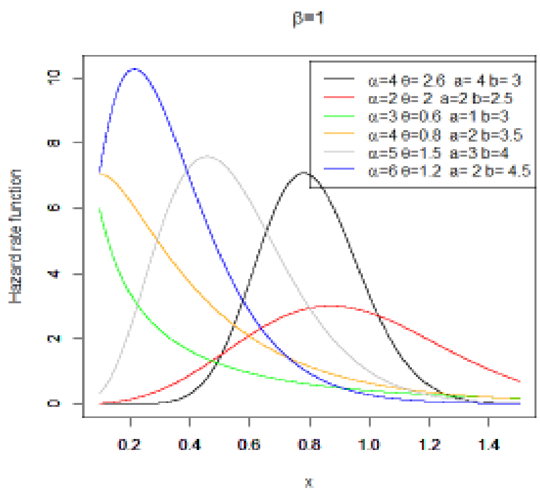

Fig. 1d: The hazard function of KGPL for some parameter values

$$(1-z)^{\{\beta-1\}} = \sum_{\{i=0\}}^{\{\infty\}} (-1)^{\{i\}\binom{\beta-1}{i}} z^{\{i\}}. \qquad (15)$$

Using (15), the Eq. (16) becomes:

$$f_{KGPL}(x,\alpha,\theta,\beta,a,b) = W_{ijk}\left(\frac{\alpha}{\alpha+1}\right)^{\{k\}}\left(\frac{ab\theta\alpha^2\beta}{1+\alpha}\right)(1+x^{\{\theta\}})x^{\{\theta(k+1)-1\}}e^{\{-\alpha(j+1)x^{\{\theta\}}\}} \qquad (16)$$

where,

$$W_{ijk} = \sum_{\{i,j=0\}}^{\{\infty\}} \sum_{\{k=0\}}^{\{j\}} \binom{j}{k} (-1)^{\{i+j\}} \binom{b-1}{i} \binom{a\beta(i+1)-1}{j}$$

STATISTICAL PROPERTIES

In this section we studied the statistical properties of the $(KGPL)$ distribution, specifically moments, moment generating function and mean residual life function.

Theorem (1): If X has KGPL $(\varphi, x), \varphi = (\alpha,\theta,\beta,a,b)$ then the r_{th} moment of X is given by the following:

$$\mu_r'(x) = W_{ijk}\left(\frac{ab\alpha^2\beta}{1+\alpha}\right)\left(\frac{\Gamma\left(\left(\frac{r}{\theta}\right)+k+1\right)}{(\alpha(j+1))^{\{\left(\frac{r}{\theta}\right)+k+1\}}}\right)\times\left[1+\left(\frac{\left(\frac{r}{\theta}\right)+k+1}{\alpha(j+1)}\right)\right]. \qquad (17)$$

Proof: Let X be a random variable following the KGPL distribution. The r_{th}ordinary moment can obtained using the well known formula:

$$\acute{\mu}_r(x) = E(X^r) = \int_0^\infty x^r f_{KGPL}(x,\alpha,\theta,\beta,a,b)\,dx = W_{ijk}\left(\frac{ab\alpha^2\beta}{1+\alpha}\right)\int_0^\infty (1+x^\theta)x^{r+\theta(k+1)-1}e^{\{-\alpha(j+1)x^\theta\}}\,dx = W_{ijk}\left(\frac{ab\alpha^2\beta}{1+\alpha}\right)\left(\frac{\Gamma\left(\left(\frac{r}{\theta}\right)+k+1\right)}{(\alpha(j+1))^{\{\left(\frac{r}{\theta}\right)+k+1\}}}\right)\times\left[1+\left(\frac{\left(\frac{r}{\theta}\right)+k+1}{\alpha(j+1)}\right)\right]. \qquad (18)$$

Which completes the proof.

The central moments μ_r and cumulants κ_r of the AWG distribution can be determined from expression (12) as $\mu_r = \sum_{m=0}^{r}(-1)^m \acute{\mu}_1\,\acute{\mu}_{r-m}$ and $\kappa_r = \acute{\mu}_{r-}\sum_{m=1}^{r-1}\binom{r-1}{m-1}k_m\,\acute{\mu}_{r-m}$, respectively, Additionally, the skewness and kurtosis can be calculated from the third and fourth standardized cumulants in the forms $SK = ((\kappa_3)/(\sqrt{(\kappa_2{}^3)}))$ and $KU = ((\kappa_4)/(\kappa_2{}^2))$, respectively.

Theorem (2): If X has (KGPL) distribution, then the moment generating function $M_X(t)$ has the following form:

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}W_{ijk}\left(\frac{ab\alpha^2\beta}{1+\alpha}\right)\left(\frac{\Gamma\left(\left(\frac{r}{\theta}\right)+k+1\right)}{(\alpha(j+1))^{\{\left(\frac{r}{\theta}\right)+k+1\}}}\right)\times\left[1+\left(\frac{\left(\frac{r}{\theta}\right)+k+1}{\alpha(j+1)}\right)\right]. \qquad (19)$$

Proof: We start with the well-known definition of the moment generating function given by:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{AWG}(x,\phi),$$

Since:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x)$$

Converges and each term is integral for all t close to 0, then we can rewrite the moment generating function as:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r) \tag{20}$$

By replacing $E(x^r)$. Hence using (20) the MGF of *(KGPL)* distribution is given by:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} W_{ijk} \left(\frac{ab\alpha^2\beta}{1+\alpha}\right) \left(\frac{\Gamma\left(\left(\frac{r}{\theta}\right)+k+1\right)}{(\alpha(j+1))^{\left\{\left(\frac{r}{\theta}\right)+k+1\right\}}}\right) \times \left[1 + \left(\frac{\left(\frac{r}{\theta}\right)+k+1}{\alpha(j+1)}\right)\right].$$

Which completes the proof.

Similarly, the characteristic function of the *KGPL* distribution becomes:

$$\varphi_X(t) = M_X(it)$$

where, $i = \sqrt{(-1)}$ is the unit imaginary number.

Conditional moments: For lifetime models, it is also of interest to obtain the conditional moments and the Mean Residual Lifetime Function (MRLF). The S_{th} conditional moment of X is given by:

$$\upsilon_s = \int_t^{\infty} x^s f_{KGPL}(x, \alpha, \theta, \beta, a, b) dx = W_{ijk} \left(\frac{ab\alpha^2\beta}{1+\alpha}\right) \int_t^{\infty} (1 + x^\theta) x^{S+\theta(k+1)-1} e^{\{-\alpha(j+1)x^\theta\}} dx$$

Setting $\alpha(j+1)x^\theta = y$, then we have:

$$\upsilon_s = W_{ijk} \left(\frac{ab\alpha^2\beta}{1+\alpha}\right) \times \left[\frac{\Gamma\left(\left(\frac{s}{\theta}\right)+k+1, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\left\{\left(\frac{s}{\theta}\right)+k+1\right\}}} + \frac{\Gamma\left(\left(\frac{s}{\theta}\right)+k+2, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\left\{\left(\frac{s}{\theta}\right)+k+2\right\}}}\right]$$

where, $\Gamma(a,t) = \int_t^{\infty} z^{\{a-1\}} e^{\{-z\}} dz$ denotes the complementary incomplete gamma function. The mean residual lifetime function is given by:

$$\begin{aligned}\mu(t) &= \upsilon_1(t) - t = E(X \mid X > t) - t \\ &= W_{ijk} \left(\frac{ab\alpha^2\beta}{1+\alpha}\right) \times \left[\frac{\Gamma\left(\left(\frac{1}{\theta}\right)+k+1, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\left\{\left(\frac{1}{\theta}\right)+k+1\right\}}} + \frac{\Gamma\left(\left(\frac{1}{\theta}\right)+k+2, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\left\{\left(\frac{1}{\theta}\right)+k+2\right\}}}\right] - t\end{aligned}$$

The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate FR function. Lifetimes can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). MRL functions that first decreases (increases) and then increases (decreases) are usually called bathtub (upside-down bathtub) shaped, BMRL (UMRL). Many authors such as Ghitany (1998), Mi (1995), Park (1985) and Tang *et al*. (1999) have been studied the relationship between the behaviors of the MRL and FR functions of a distribution.

RESIDUAL LIFE AND REVERSED FAILURE RATE FUNCTION

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X - t|X > t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely (Gupta and Kundu, 1999). Therefore, we obtain the r^{th} order moment of the residual life via the general formula:

$$\mu_r(t) = E((X-t)^r \mid X > t) = \left(\frac{1}{F(t)}\right) \int_r^{\infty} (x-t)^r f(x,\phi) dx, r \geq 1. \tag{21}$$

Applying the binomial expansion of $(x-t)^r$ and substituting $f(x,\phi)$ given by (16) into the above formula gives:

$\mu_r(t) = \frac{W_{ijk}ab\theta\alpha^2\beta}{\bar{F}(t)} \sum_{i=0}^{r}(-1)^i\binom{r}{i} \times \int_t^\infty (1+x^\theta)x^{r+\theta(k+1)-1}e^{\{-\alpha(j+1)x^\theta\}}\,dx\mu_r(t) =$

$$\frac{W_{ijk}ab\theta\alpha^2\beta}{\bar{F}(t)} \sum_{i=0}^{r}(-1)^i\binom{r}{i} \times \left[\frac{\Gamma\left(\left(\frac{r-i}{\theta}\right)+k+1, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\{\left(\frac{1}{\theta}\right)+k+1\}}} + \frac{\Gamma\left(\left(\frac{r-i}{\theta}\right)+k+2, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\{\left(\frac{s}{\theta}\right)+k+2\}}}\right] \quad (22)$$

where, $\Gamma(s,t) = \int_t^\infty x^{\{s-1\}}e^{\{-x\}}dx$ is the upper incomplete gamma function.

On the other hand, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable $t - X|X \leq t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable may also be called the inactivity time (or time since failure. Also, in reliability, the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely. The r^{th}order moment of the reversed residual life can be obtained by the well known formula:

$$m_r(t) = E((t-X)^r \mid X \leq t) = \left(\frac{1}{F(t)}\right)\int_0^t (t-x)^r f(x,\phi)dx, r \geq 1. \quad (23)$$

Applying the binomial expansion of $(t-x)^r$ and substituting $f(x,\phi)$ given by (16) into the above formula gives:

$$m_r(t) = \frac{W_{ijk}ab\theta\alpha^2\beta}{F(t)} \sum_{i=0}^{r}(-1)^i\binom{r}{i} \times \int_t^\infty (1+x^\theta)x^{r+\theta(k+1)-1}e^{\{-\alpha(j+1)x^\theta\}}\,dxm_r(t)$$
$$= \frac{W_{ijk}ab\theta\alpha^2\beta}{\bar{F}(t)} \sum_{i=0}^{r}(-1)^i\binom{r}{i}$$
$$\times \left[\frac{\gamma\left(\left(\frac{r-i}{\theta}\right)+k+1, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\{\left(\frac{1}{\theta}\right)+k+1\}}} + \frac{\gamma\left(\left(\frac{r-i}{\theta}\right)+k+2, \alpha(j+1)t^\theta\right)}{(\alpha(j+1))^{\{\left(\frac{s}{\theta}\right)+k+2\}}}\right]$$

where $\gamma(s,t) = \int_t^\infty x^{s-1}e^{-x}dx$is the lower incomplete gamma function.

ESTIMATION: Estimation of the model parameters of the kumaraswamy generalized power Lindley distribution can be accomplished by the maximum likelihood method. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from KGPL $(\varphi, x), \varphi = (\alpha, \theta, \beta, a, b)$. Let $\varphi = (\alpha, \theta, \beta, a, b)^{\{T\}}$ be the parameter vector. The log likelihood function for the vector of parameters $\varphi = (\alpha, \theta, \beta, a, b)$ can be written as:

$$\log L = nlog\ a + nlog\ b + nlog\theta + 2nlog\alpha - nlog(1+\alpha) + \sum_{i=1}^{n}\log(1+x_i^\theta) + (\theta-1)\sum_{i=1}^{n}\log(x_i) + \alpha\sum_{i=1}^{n}(x_i)^\theta + (\beta a - 1)\sum_{i=1}^{n}\log\left[1-\left(1+\left(\frac{\alpha x_i^\theta}{\alpha+1}\right)\right)e^{\{-\alpha x_i^\theta\}}\right] + (b-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left(1+\left(\frac{\alpha x_i^\theta}{\alpha+1}\right)\right)e^{\{-\alpha x_i^\theta\}}\right]^{\{a\beta\}}\right] \quad (24)$$

The associated score function is given by:

$$U_n(\varphi) = \left[\left(\frac{\partial L}{\partial \alpha}\right), \left(\frac{\partial L}{\partial \theta}\right), \left(\frac{\partial L}{\partial \beta}\right), \left(\frac{\partial L}{\partial a}\right), \left(\frac{\partial L}{\partial b}\right)\right]^T$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (24). The components of the score vector are given by:

$$\frac{\partial logL}{\partial \alpha} = \left(\frac{2n}{\alpha}\right) - \left(\frac{n}{1-\alpha}\right) - \sum_{i=1}^{n}(x_i)^\theta + (\beta a - 1) \times \sum_{i=1}^{n}\frac{\frac{e^{\{-\alpha x_i^\theta\}}}{(\alpha+1)^2} - x_i^\theta\omega(x_i)}{[1-\omega(x_i)]} + a\beta(b-1) \times \sum_{i=1}^{n}\frac{[1-\omega(x_i)]^{a\beta-1}\left[\frac{e^{\{-\alpha x_i^\theta\}}}{(\alpha+1)^2} - x_i^\theta\omega(x_i)\right]}{1-[1-\omega(x_i)]^{a\beta}} \quad (25)$$

$$\frac{\partial logL}{\partial\theta} =$$

$$\frac{n}{\theta}-\sum_{i=1}^{n}\frac{x_i^{\theta}\log(x_i)}{\left(1+x_i^{\theta}\right)}+\sum_{i=1}^{n}lo\,g(x_i)-\alpha\sum_{i=1}^{n}(x_i)^{\theta}\,lo\,g(x_i)+(\beta a-1)\times\sum_{i=1}^{n}\frac{\alpha\omega(x_i)x_i^{\theta}lo\,g(x_i)-\frac{\alpha x_i^{\theta}}{\alpha+1}e^{\left\{-\alpha x_i^{\theta}\right\}}lo\,g(x_i)}{1-\omega(x_i)} \quad (26)$$

$$\frac{\partial logL}{\partial\beta}=a\sum_{i=1}^{n}lo\,g(1-\omega(x_i))-a(b-1)\times\sum_{i=1}^{n}\frac{[1-\omega(x_i)]^{a\beta}log[1-\omega(x_i)]}{1-[1-\omega(x_i)]^{a\beta}} \quad (27)$$

$$\frac{\partial logL}{\partial a}=\frac{n}{a}-\beta\sum_{i=1}^{n}lo\,g(1-\omega(x_i))-\beta(b-1)\times\sum_{i=1}^{n}\frac{[1-\omega(x_i)]^{a\beta}log[1-\omega(x_i)]}{1-[1-\omega(x_i)]^{a\beta}} \quad (28)$$

And

$$\frac{\partial logL}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}log\left[1-[1-\omega(x_i)]^{a\beta}\right]. \quad (29)$$

where,

$$\omega(x_i)=\left(1+\frac{\alpha x_i^{\theta}}{\alpha+1}\right)e^{\left\{-\alpha x_i^{\theta}\right\}}$$

And The Maximum Likelihood Estimation (MLE) of ϕ, say $\hat{\phi}$, is obtained by solving the nonlinear system $U_n(\phi)=0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton--Raphson type algorithm to obtain the estimate$\hat{\phi}$. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 5×5 observed information matrix is given by:

$$(\varphi)=-\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\theta} & I_{\alpha\beta} & I_{\alpha a} & I_{\alpha b}\\ I_{\theta\alpha} & I_{\theta\theta} & I_{\theta\beta} & I_{\theta a} & I_{\theta b}\\ I_{\beta\alpha} & I_{\beta\theta} & I_{\beta\beta} & I_{\beta a} & I_{\beta b}\\ I_{a\alpha} & I_{a\theta} & I_{a\beta} & I_{aa} & I_{ab}\\ I_{b\alpha} & I_{b\theta} & I_{b\beta} & I_{ba} & I_{bb}\end{pmatrix}$$

Whose elements are given in Appendix. Applying the usual large sample approximation, MLE of ϕ, i.e., $\hat{\phi}$ can be treated as being approximately $N_5(\phi, J_n(\phi)^{-1})$, where $J_n(\phi)=E[I_n(\phi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\phi}-\phi)$ is $N_5(0, J(\phi)^{-1})$, where $J(\phi) = lim\_\{n\to\infty\}n^{-1}I_n(\phi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\phi)$ is replaced by the average sample information matrix evaluated at $\hat{\phi}$, say $n^{-1}I_n(\hat{\phi})$. The estimated asymptotic multivariate normal $N_5(\phi, I_n(\hat{\phi})^{-1})$ distribution of ϕ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for each parameter ϕ_r is given by:

$$ACI_r=(\phi_r - z_{\frac{\gamma}{2}}\sqrt{\hat{I}_{rr}}\hat{\phi}_r+z_{\frac{\gamma}{2}}\sqrt{\hat{I}_{rr}})$$

where, z_γ is the upper 100γ the percentile of the standard normal distribution.

APPLICATIONS

In this section, we present examples that illustrate the flexibility and the applicability of the KGPL distribution in modelling real world data. We t the density functions of the KGPL distribution and the GPL. We also compare the KGPL to other comparable distributions. For each data set, the estimates of the parameters of the distributions and information criterion statistics are calculated.

Cancer patients data set: The first data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang (2013). Estimates of the parameters of the KGPL distribution, Akaike information criterion (AIC), consistent Akaike information criterion (AICC), Bayesian information criterion (BIC) are given in Table 1 for cancer patients data.

Table 1: MLEs, LSE, the measures AIC, AICC and BIC and KS test to Cancer Patients data

Model	Estimates	-logL	AIC	AICC	BIC	KS
KGPL	$\hat{a} = 56.12$	411.5	833.0	833.5	847.3	0.042
	$\hat{b} = 240.2$					
	$\hat{\alpha} = 6.309$					
	$\hat{\beta} = 43.11$					
	$\hat{\theta} = 0.030$					
TEMW	$\hat{\alpha} = 0.254$	463.2	936.4	936.9	950.7	0.735
	$\hat{\beta} = 0.073$					
	$\hat{\gamma} = 0.0006$					
	$\hat{\theta} = 0.009$					
	$\hat{\lambda} = 0.033$					
KPL	$\hat{a} = 1.313$	416.8	841.7	842.0	848.0	0.090
	$\hat{b} = 0.001$					
	$\hat{\beta} = .0244$					
	$\hat{\theta} = 0.051$					
L	$\hat{\theta} = 0.196$	419.5	841.0	841.0	843.9	0.11

Table 2: MLEs, LSE, the measures AIC, AICC and BIC and KS test to Guinea Pigs data

Model	Estimates	-logL	AIC	AICC	BIC	KS
KGPL	$\hat{a} = 70.59$	94.80	199.6	200.5	210.9	0.08
	$\hat{b} = 281.4$					
	$\hat{\alpha} = 6.753$					
	$\hat{\beta} = 43.11$					
	$\hat{\theta} = 43.11$					
TEMW	$\hat{\alpha} = 0.201$	115.2	240.5	241.4	251.9	0.79
	$\hat{\beta} = 0.193$					
	$\hat{\gamma} = 0.256$					
	$\hat{\theta} = 1.64$					
	$\hat{\lambda} = 0.343$					
E	$\hat{a} = 0.565$	113.0	228.0	228.1	230.3	0.28
L	$\hat{\theta} = 0.86$	106.9	215.8	215.9	218.1	0.23

Guinea pigs data set: The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The starting point of the iterative processes for the guinea pigs data set is (1:0; 0:009; 10:0; 0:1; 0:1).

The values in Table 2 indicate that the *KGPL* distribution leads to a better fit over all the other models (Fig. 2 and 3).

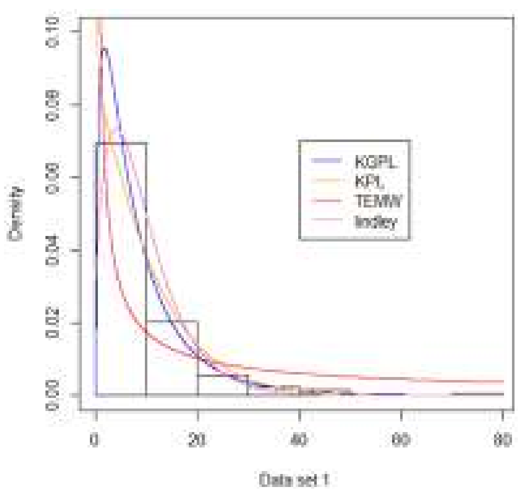

Fig. 2a: Estimated densities of the KGPL, TEMW, KPL and L distributions for the data

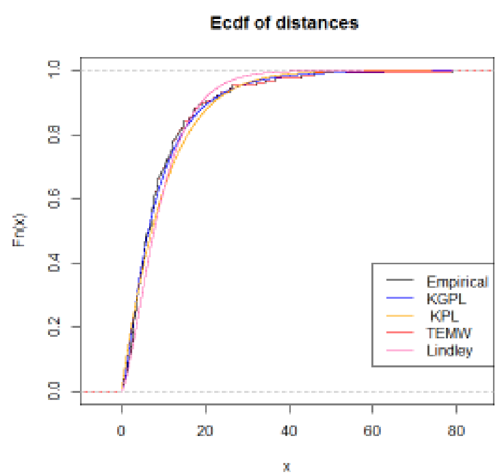

Fig. 2b: Estimated cdf function from the fitted the KGPL, TEMW, KPL and L distributions and the empirical cdf of the data set 1

CONCLUSION

There has been an extraordinary enthusiasm among statisticians and connected specialists in developing adaptable lifetime models to encourage better demonstrating of survival information. Hence, a huge advancement has been made towards the speculation of some surely understood lifetime models and their fruitful application to issues in a few ranges. In this

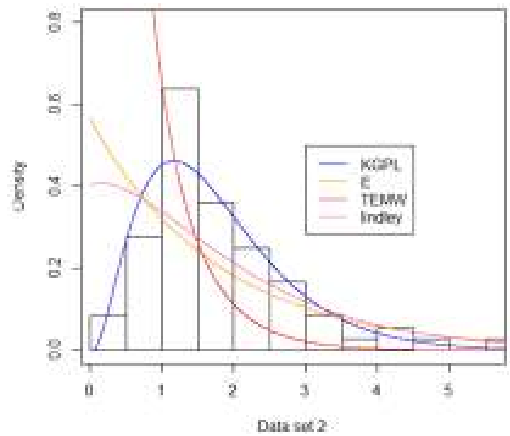

Fig. 3a: Estimated densities of the KGPL, TEMW, E and L distributions for the data

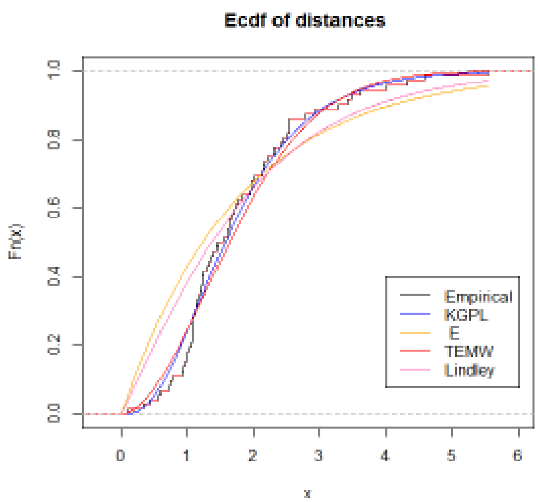

Fig. 3b: Estimated cdf function from the fitted the KGPL, TEMW, E and L distributions and the empirical cdf of the data set 1

model study, we present another four-parameter got utilizing the Kumaraswamy generalization technique. We refer to the new model as the KGPL distribution and study some of its mathematical and statistical properties. We hope that the proposed distribution will serve as an alternative model to other models available in the literature.

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